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# THE COMPARATIVE STATICS FOR LINEAR PAYOFFS AND INCREASES IN RISK*

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*This paper introduces a concept for the subset of K-L-L-S increases in risk defined by Kroll, Leshno, Levy, and Spector (1995; called K-L-L-S); a 'relatively strong increase in risk in the K-L–L-S sense' ($RSIR_K$). Our new notion of K-L-L-S increases in risk extends the Rothschild-Stiglitz definition of risk to a larger set of cumulative distribution functions, but use somewhat stronger restrictions on the structure of the decision model and the set of decision-makers. The decision model used in this paper consists of a utility function of one scalar variable that is affected by one-dimensional choice variable (and another random variable) to avoid problems involving multidimensionality. We show that, by restricting the payoff function to be linear in the random variable ( $z_{xx} = 0$ ) and limiting our analysis to decision-makers who are prudnet ( $u''' \geq 0$ ), we are able to generate comparative statics results for the $RSIR_K$ order.*

JEL Classification: D81
Keywords: Kroll, Leshno, Levy, and Spector (K-L-L-S) increases in risk, Relatively strong increases in risk, Relatively strong increases in risk in the K-L-L-S sense, Linear payoffs, Prudence

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## I. INTRODUCTION

There are several studies concerning the comparative statics analysis considering specified particular types of cumulative distribution function (CDF) changes which are subsets of second-degree stochastic dominance (SSD) shifts or Rothschild-Stiglitz increases in risk defined by Rothschild and Stiglitz (1970). For examples, to obtain the comparative statics results with subsets of Rothschild-Stiglitz increases in risk, Meyer and Ormiston (1985) defined a 'strong increase in risk' (SIR). Black and Bulkley (1989) introduced the concept of a 'relatively strong increase in risk' (RSIR.). Dionne, Eeckhoudt and Gollier (1993a, 1993b) considered a 'relatively weak increase in risk' (RWIR). The relationship among these increases in risk is that the SIR order implies the RSIR order which, in turn, implies the RWIR order. Recently Ryu and Kim (2004a, 2004b) introduced the concept of a 'left-side strong increase in risk' (L-SIR) order and a 'left-side relatively weak increase in risk' (L-RWIR) order as the subset of Rothschild-Stiglitz increases in risk that extends the definition of strong increases in risk introduced by Meyer and Ormiston and of relatively weak increases in risk established by Dionne, Eeckhoudt and Gollier.

However, no one has examined the subsets of third-degree stochastic dominance (TSD) shifts for comparative statics purposes except for Ryu and Kim (2005). Kroll, Leshno, Levy, and Spector (1995; named K-L-L-S) defined a special 'increase in risk' as an TSD change with equal means. More recently Ryu and Kim (2005) proposed the concept of a simple increases in risk across $r$ in the K-L-L-S sense (sIR($r$)) for the subset of K-L-L-S increases in risk and provided comparative statics results in the standard portfolio decision problem. Note that whereas Rothschild-Stiglitz increases in risk are SSD changes with equal means, K-L-L-S increases in risk are TSD changes with equal means. Since SSD implies TSD, the set of K-L-L-S increases in risk includes the set of Rothschild-Stiglitz increases in risk.

In this paper, we propose the new definition for the subset of K-L-L-S increases in risk named a 'relatively strong increase in risk in the K-L-L-S sense' ($RSIR_K$). Since an K-L-L-S increase in risk is a particular type of TSD change, the derived comparative statics statements are associated with a risk-averse decision-maker with $u''' \geq 0$. We consider the sets of the utility function representing quite plausible preferences, such as decreasing absolute risk aversion (DARA) implying 'prudence' $(\eta = -u'''/u'')$ defined by Kimball (1990), which denotes the intensity of a precautionary saving motive. Note that the term 'prudence' is meant to suggest the propensity to prepare and forearm oneself in the face of uncertainty. Therefore, this paper focuses on the sufficient condition on the change in distribution of the random parameter that causes risk-averse decision makers with $u''' \geq 0$ to adjust their choice variable in the same direction in a general decision model.

This paper provides comparative statics results regarding the subset of increases in risk in the K-L-L-S sense and compares our results with the results for the subset of increases in risk in the Rothschild-Stiglitz sense termed a relatively strong increase in risk. This paper shows that K-L-L-S increases in risk extend the Rothschild-Stiglitz definition of risk to a larger set of CDFs that could not be classified as 'more risky' before.

This paper is organized in the following way. In section II, we present a model of a decision-maker maximizing the expected utility and provide the definition for the subset of K-L-L-S increases in risk and graphical example. In section III, we give comparative statics results concerning K-L-L-S increases in risk. Finally, section IV provides concluding remarks.

## II. THE MODEL AND SUBSET OF INCREASES IN RISK IN THE K-L-L-S SENSE

We analyze a model of a payoff function $z(x,\alpha)$, where $x$ is a random variable and $\alpha$ is a choice variable. A decision-maker selects $\alpha$ in order to maximize his expected utility $E(u[z(x,\alpha)])$, where utility depends only on the payoff function $z(x,\alpha)$, which is a scalar valued function of one choice variable and one random variable. According to Meyer and Ormiston (1985), this type of formulation is previously employed by Kraus (1979) and Katz (1981) and has several advantages such that problems involving multidimensionality are avoided and in the measure of absolute and relative risk aversion considered by Pratt (1964) and Arrow (1971) are readily calculated. We assume that utility function $u(z)$ is thrice differentiable $u'(z) \geq 0$, $u''(z) < 0$, and $u'''(z) \geq 0$, where the assumption of $u'''(z) \geq 0$ is interpreted as aversion or neutrality to an increase in downside risk proposed by Menezes et al. (1980).

In comparative statics analysis, we assume that the supports of $F(x)$ and $G(x)$ are located on the interval $(x_2, x_4)$ and $(x_1, x_3)$, respectively, where $x_1 \le x_2 \le x_3 \le x_4$, and $f(x)$ and $g(x)$ are the probability density functions (PDFs) of $F(x)$ and $G(x)$, respectively. We define that $\hat{F}(x) = \int_{x_1}^{x} F(t)\,dt$ and $\hat{G}(x) = \int_{x_1}^{x} G(t)\,dt$. We also assume that the first- and the second-order condition are satisfied to guarantee a unique interior solution. Faced by the CDF of the random variable $F(x)$, the first-order condition defining the optimal value for the choice of $\alpha$ to maximize expected utility is

$$\int_{x_2}^{x_4} u'(z) z_\alpha(x, \alpha_F)\, dF(x) = 0. \tag{1}$$

The optimal solution satisfying (1) is guaranteed to be a global optimum by the second-order condition that

$$\int_{x_2}^{x_4} \left\{ u''[z(x,\alpha_F)]\, z_\alpha^2 + u'[z(x,\alpha_F)]\, z_{\alpha\alpha} \right\} dF(x) < 0. \tag{2}$$

It is assumed that $u''(z) < 0$ and $z_{\alpha\alpha} \le 0$. Thus, in order to see the result that $\alpha_F \ge \alpha_G$ for a specified change in the CDF from *F* to *G*, it is sufficient to show that

$$Q(\alpha_F) = \int_{x_1}^{x_4} u'[z(x,\alpha_F)]\, z_\alpha(x,\alpha_F)\, d[F(x) - G(x)] \ge 0. \tag{3}$$

Kroll et al. (1995) proposed a new concept of probability mass shifts called a ‘mean preserving spread-antispread’. We called it an ‘increase in risk in the K-L-L-S sense’.

**Definition 1:** $G(x)$ is said to be riskier than $F(x)$ in the K-L-L-S sense if and only if

(a) $\int_{x_1}^{x_4} [G(x) - F(x)]\, dx = 0$

(b) $\int_{x_1}^{s}\left[\hat{G}(x)-\hat{F}(x)\right]dx \geq 0$ for all $s \in [x_1, x_4]$.

Condition (a) implies that two distributions have equal means. Condition (b) implies that a mean preserving spread-antispread function satisfies the TSD criterion. These conditions imply that an increase in risk in the K-L-L-S sense is an TSD change with equal means. Since SSD implies TSD, the set of K-L-L-S increases in risk includes the set of Rothschild-Stiglitz increases in risk. Observe that, for random variables with equal means, Definition 1 is equivalent to the TSD rule.

Black and Bulkley (1989) introduced the concept of a 'relatively strong increase in risk' (RSIR) that is a particular type of R-S increases in risk by imposing the restriction on the ratio of a pair of PDFs. By replacing the restriction on the ratio of a pair of PDFs with the restriction on the ratio of a pair of CDFs, we now introduce the new definition for the subset of increases in risk in the K-L-L-S sense and call it a 'relatively strong increase in risk in the K-L-L-S sense'.

**Definition 2:** $G(x)$ represents a relatively strong increase in risk in the K-L-L-S sense with respect to $F(x)$ ( denoted by $G$ RSIR$_K$ $F$ ) if

(a) $\int_{x_1}^{x_4}\left[G(x)-F(x)\right]dx = 0$

(b) $\int_{x_1}^{s}\left[\hat{G}(x)-\hat{F}(x)\right]dx \geq 0$ for all $s \in [x_1, x_4]$

(c) There exists a point $m^0$ satisfying $\int_{x_1}^{m^0}\left[G(x)-F(x)\right]dx = \int_{m^0}^{x_4}\left[G(x)-F(x)\right]dx = 0$

(d) There exists a pair of points $m_1, m_2 \in [x_2, x_3]$ where $m_1 \leq m_2$, such that $F(x) \geq G(x)$ for all $x \in [m_1, m_2]$ and $F(x) \leq G(x)$ for all $x$ in $[x_2, m_1]$ and $[m_2, x_3]$

(e) For all $x \in [x_2, m_1]$, there exists a non-decreasing function $H_1 : [x_2, m_1) \to [0,1]$ such that $F(x) = H_1(x)G(x)$

(f) For all $x \in [m_2, x_3]$, there exists a non-increasing function $H_2 : (m_2, x_3] \to [0,1]$ such that $F(x) = H_2(x)G(x)$.

Conditions (a) and (b) define an K-L-L-S increase in risk. Condition (c) means that a relatively strong increase in risk in the K-L-L-S sense satisfies second-degree stochastic dominance (SSD) on both side of $m_0$ with equal means. Condition (d) imposes the restriction that the two CDFs cross only twice. Conditions (e) and (f) restrict the extent to which cumulative probability mass can be transferred to any one value in the tails of $F(x)$ relative to any other and impose monotonicity restrictions on the ratio between a pair of CDFs on the intervals $x \in [x_2, m_1]$ and $x \in [m_2, x_3]$. Note that conditions (c) to (f) are added to identify this particular type of increases in risk in the K-L-L-S sense and allow comparative static analysis to be made.

**[Figure 1]** *G* $\text{RSIR}_K$ *F*

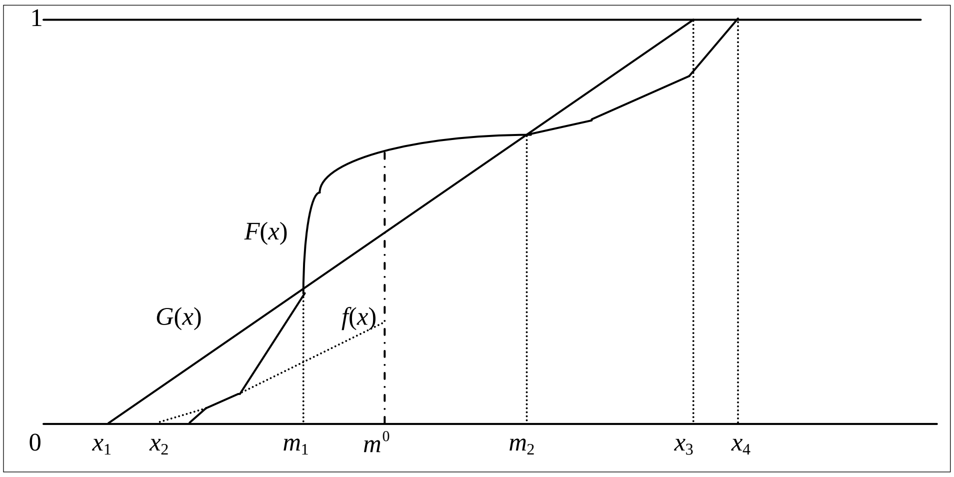



Figure 1 satisfies the $\text{RSIR}_K$ conditions given in Definition 2. $H_1$ is non-decreasing and $H_2$ is non-increasing for each corresponding interval $x \in [x_2, m_1]$ and $x \in [m_2, x_3]$, respectively. If, for any point on the interval $[x_2, m_1)((m_2, x_3])$, the tangent lines $f(x)$ and $g(x)$ do not meet to the left (right) direction in the probability space between zero and one, the ratio of a pair of CDFs $H_1(H_2)$ is non-decreasing (non-increasing). Note that an $\text{RSIR}_K$ order consists of the sum of three shifts, which are a 'left-side monotone probability ratio' (L-MPR) on the interval $x \in [x_1, m_1]$, a 'right-side monotone probability ratio' (R-MPR) on the interval $x \in [m_2, x_4]$ and a first-order stochastically dominated shift (the opposite of an FSD shift) on the interval $x \in [m_1, m_2]$.

## III. COMPARATIVE STATICS ANALYSIS

In this section, we provide comparative statics results for a relatively strong increase in risk in the K-L-L-S sense. We assume that $z_x > 0$, $z_{\alpha x} \geq 0$, and $z_{xx} = 0$. Since (3) is changed as (4), given these assumptions about $z(x,\alpha)$ to prove $\alpha_F \geq \alpha_G$ it is sufficient to show

$$Q(\alpha_F) = \int_{x_1}^{x_4} u'(z)\, z_\alpha(x,\alpha_F)\, d(F(x) - G(x)) \geq 0. \tag{4}$$

Integration by parts of the RHS of $Q(\alpha_F)$ in (4) yields

$$\begin{aligned} Q(\alpha_F) &= u'(z(x_4,\alpha_F))[T(x_4,\alpha_F;F,z) - T(x_4,\alpha_F;G,z)] \\ &\quad - \int_{x_1}^{x_4} u''(z)\, z_x [T(x,\alpha_F;F,z) - T(x,\alpha_F;G,z)]\, dx \geq 0 \end{aligned} \tag{5}$$

where $T(\cdot,\cdot;I,z):[x_1,x_4]\times R \rightarrow R$ is defined as

$$T(x,\alpha_F;I,z) \equiv \int_{x_1}^{x} z_\alpha(t,\alpha_F)\, dI(t) \qquad \text{where } I = F, G.$$

where $T$ denotes location-weighted probability mass function used in Gollier (1995).

In order to prove Theorem, we need first to introduce the following Lemmas 1 and 2.

**Lemma 1:** If $z_{\alpha x} \geq 0$, then $\varphi(s) = \int_{x_2}^{s} u''(z)\, z_x T(x,\alpha_F;F,z)\, dx \geq 0$ for all $s$ in $[x_2, x_4]$.

**Proof:** Note that $\varphi(x_2) = 0$. By integrating $\int_{x_2}^{x_4} u'(z)\, z_\alpha(x,\alpha_F)\, dF(x)$ by parts, we get $\varphi(s) = u'(z)T(s,\alpha;F,z) - \int_{x_2}^{s} u'(z)\, z_\alpha dF(x)$. From the first-order condition (1) and the assumptions $u' \geq 0$ and $z_{\alpha x} \geq 0$, we obtain

$\int_{x_2}^{s} u'(z) z_\alpha dF(x) \leq 0$ for all $s$ in $[x_2, x_4]$. Since the sign of $\frac{\partial T}{\partial x}(x, \alpha; F, z)$ is equal to the sign of $z_\alpha$ $(z_{\alpha x} \geq 0)$, $T$ must alternate in sign. Let $x^c$ denote the value of $x$ where $T$ changes sign from negative to positive. For all $s \geq x^c$, $\varphi(s) \geq 0$ because the first term is positive and the second one with minus sign is also. For all $s \leq x^c$, $\varphi(s)$ is an increasing function because $u''(z) z_x T(x, \alpha; F, z) \geq 0$. Given that $\varphi(x_2) = 0$, $\varphi(s) \geq 0$ for all $s \leq x^c$.

Q.E.D.

Before proving Lemma 2, we need to know the following things. The integrand in expression (5), $u''(z) z_x [T(x, \alpha_F; F, z) - T(x, \alpha_F; G, z)]$, has a sign depending on the difference between two $T$ functions.

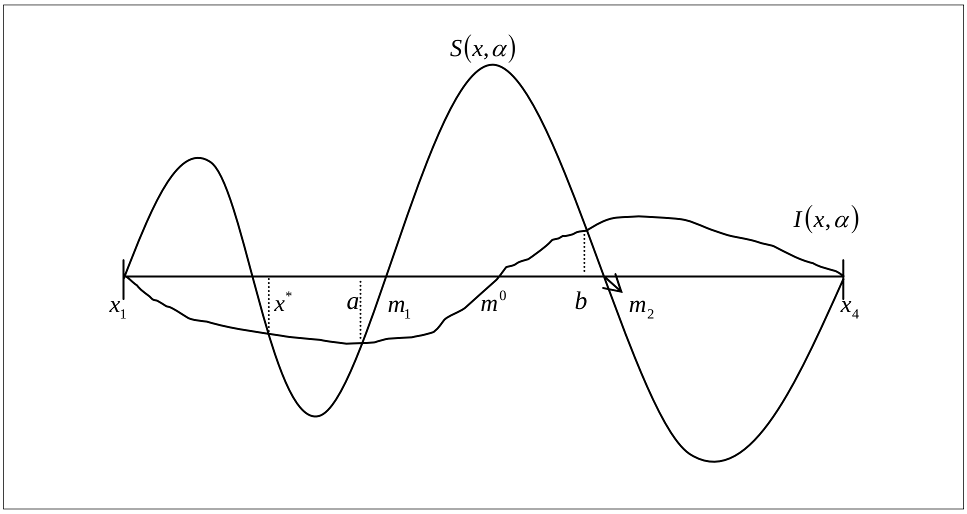
$$T(x, \alpha_F; F, z) - T(x, \alpha_F; G, z)$$
$$= z_\alpha [F(x) - G(x)] - \int_{x_1}^{x} z_{\alpha t} [F(t) - G(t)] dt . \tag{6}$$

The $z_\alpha$ has a sign change to satisfy the first order condition (1) and the condition $z_{\alpha x} \geq 0$. When it changes sign in one of the intervals $[x_1, m_1]$, $[m_1, m^0]$, $[m^0, m_2]$ and $[m_2, x_4]$, (6) changes sign at most two times from positive to negative for the interval $[x_1, x_4]$ (See Figures 2-5). When (6) has a sign change only once, it is trivial. Over four intervals let $x^*$ denote the point where (6) changes its sign when $z_\alpha$ has a sign change for each interval. Let us define that $S(x, \alpha) = z_\alpha [F(x) - G(x)]$ and $I(x, \alpha) = \int_{x_1}^{x} z_{\alpha t} [F(t) - G(t)] dt$. For example, observe that in Figure 2, $x^*$ is the point where $S(x, \alpha) = I(x, \alpha)$ when the sign of $z_\alpha$ changes from negative to positive ( $z_{\alpha x} \geq 0$ ) for the interval $[x_1, m_1]$..

Let $k_1$, $k_2$, $k_3$ be the ordered values of $x^*$, $a$, $b$(See case (i) – case (iii)). We consider the following four cases in (6); (i) $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ (ii-a) and (ii-b) $k_1 = a$, $k_2 = x^*$ and $k_3 = b$, and (iii) $k_1 = a$, $k_2 = b$ and $k_3 = x^*$, where $k_2$ is the end point when the first sign change occurs for each interval, and $k_1$ and $k_3$ are the points which the first and the second sign change occurs from positive to negative, respectively. Now, we draw the figures to get a clear demonstration for each case.

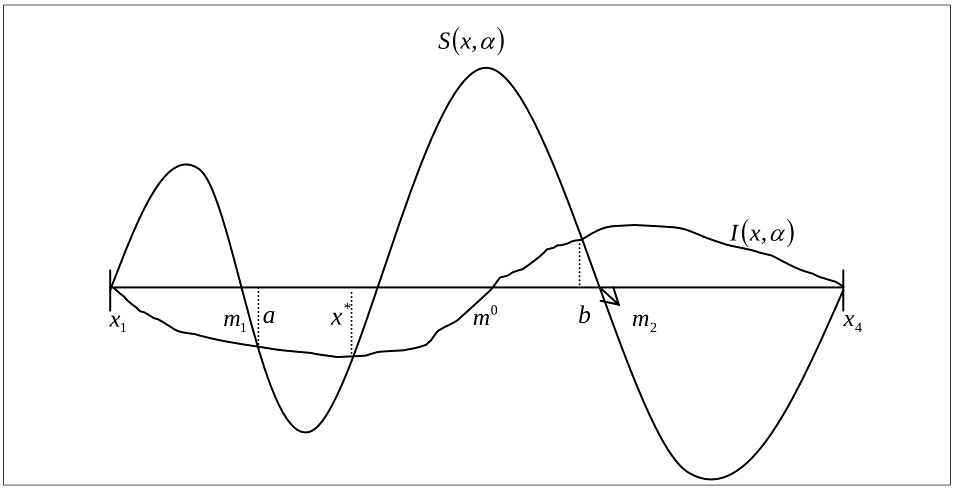
Case (i): $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ when $x_1 \le x^* \le m_1$.

**[Figure 2]** $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ when $x_1 \le x^* \le m_1$



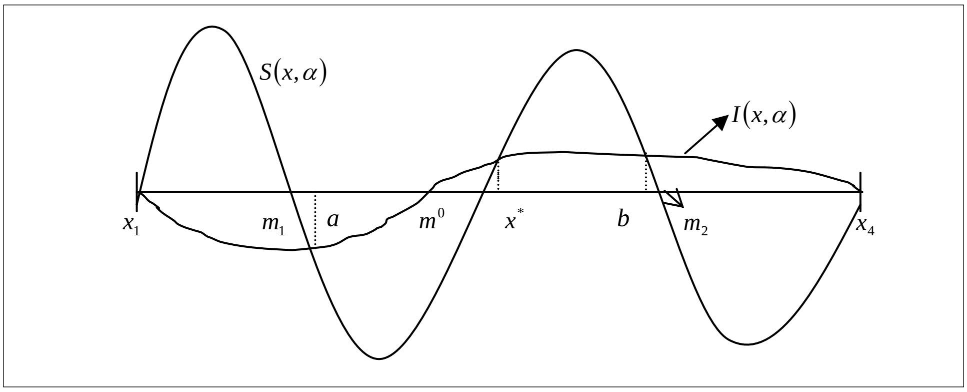

Case (ii-a): $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m_1 \le x^* \le m^0$.

**[Figure 3]** $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m_1 \le x^* \le m^0$



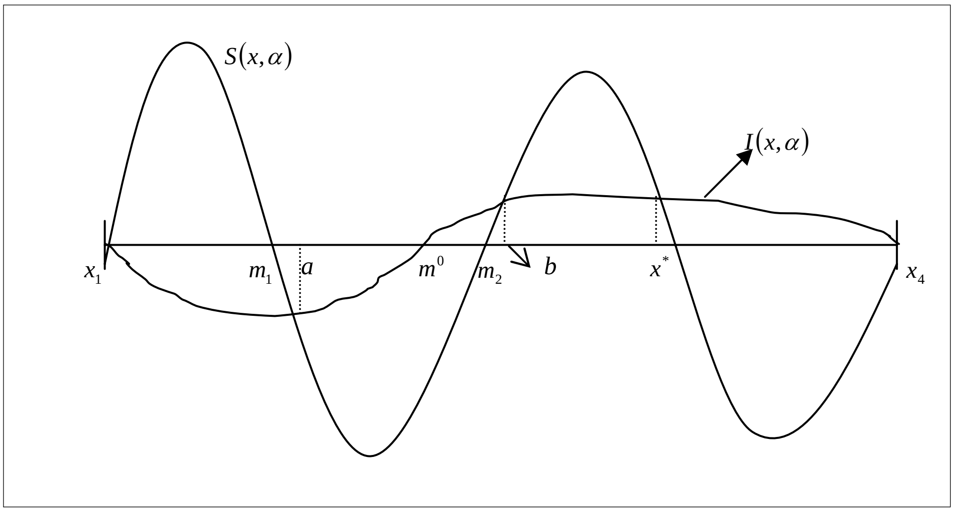

Case (ii-b): $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m^0 \le x^* \le m_2$.

**[Figure 4]** $k_1 = a$ , $k_2 = x^*$ and $k_3 = b$ when $m^0 \le x^* \le m_2$




Case (iii): $k_1 = a$ , $k_2 = b$ and $k_3 = x^*$ when $m_2 \le x^* \le x_4$ .

**[Figure 5]** $k_1 = a$ , $k_2 = b$ and $k_3 = x^*$ when $m_2 \le x^* \le x_4$ .




**Lemma 2:** The value of α which maximizes $E[u(z(x,\alpha))]$ is lower for a class of risk-averse decision-makers with $u''' \ge 0$ under $G(x)$ than $F(x)$ where $G(x)$ and $F(x)$ satisfy conditions (a), (b) and (c) for $G(x)$ to represent a $RSIR_K$ presented in Definition 1 if $\int_{x_1}^{k_2} [T(x,\alpha_F;F,z) - T(x,\alpha_F;G,z)]dx > 0$ and if the given assumptions about $z(x,\alpha)$ are satisfied, where $k_2$ is the end point when the first sign change occurs for each interval, and $k_1$ and $k_3$ are the points which the first and the second sign change occurs from positive to

negative, respectively in (6).

**Proof:** For notational convenience, we define that $T(x,\alpha_F;F,z)= T_F(x,\alpha_F)$ and $T(x,\alpha_F;G,z)=T_G(x,\alpha_F)$.

$$Q(\alpha_F)=u'(z(x_4,\alpha_F))[T_F(x_4,\alpha_F)-T_G(x_4,\alpha_F)] \\ -\int_{x_1}^{x_4}u''(z)\,z_x[T_F(x,\alpha_F)-T_G(x,\alpha_F)]dx\,.$$

Using these assumptions about $z(x,\alpha)$ and the RSIR$_K$ definition, the first term of $Q(\alpha_F)$ by integrating of $[T_F(x_4,\alpha_F)-T_G(x_4,\alpha_F)]$ by parts is non-negative. Therefore,

$$Q(\alpha_F)\geq -\int_{x_1}^{k_2}u''(z)\,z_x[T_F(x,\alpha_F)-T_G(x,\alpha_F)]dx\,. \\ -\int_{k_2}^{x_4}u''(z)\,z_x[T_F(x,\alpha_F)-T_G(x,\alpha_F)]dx\,.$$

Since $-u''(z)\,z_x$ is non-negative and decreasing and $[T_F-T_G]$ has a sign change, we have the following four cases:

(i) $k_1=x^*$, $k_2=a$ and $k_3=b$ (ii-a) and (ii-b) $k_1=a$, $k_2=x^*$ and $k_3=b$, and (iii) $k_1=a$, $k_2=b$ and $k_3=x^*$.

$$Q(\alpha_F)\geq -u''(z(k_1,\alpha_F))\,z_x\int_{x_1}^{k_2}[T_F(x,\alpha_F)-T_G(x,\alpha_F)]dx \\ -u''(z(k_3,\alpha_F))\,z_x\int_{k_2}^{x_4}[T_F(x,\alpha_F)-T_G(x,\alpha_F)]dx\,.$$

Adding and subtracting $-u''(z(k_3,\alpha_F))\,z_x\int_{x_1}^{k_2}[T_F(x,\alpha_F)-T_G(x,\alpha_F)]dx$ on the RHS of the above inequality gives

$$Q(\alpha_F)\geq \{-u''(z(k_1,\alpha_F))\,z_x+u''(z(k_3,\alpha_F))\,z_x\}\int_{x_1}^{k_2}[T_F(x,\alpha_F)-T_G(x,\alpha_F)]dx \\ -u''(z(k_3,\alpha_F))\,z_x\int_{x_1}^{x_4}[T_F(x,\alpha_F)-T_G(x,\alpha_F)]dx\,.$$

Since $-u''(z)\,z_x$ is non-negative and decreasing, the bracket of the first term on the RHS of the above inequality is non-negative.

If $\int_{x_1}^{k_2}\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx \ge 0$,

$$Q(\alpha_F) \ge -u''(z(k_3,\alpha_F))\,z_x \int_{x_1}^{x_4}\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx.$$

Integrating of $\int_{x_1}^{x_4}\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]$ by parts and using the assumptions about $z(x,\alpha)$ and the $RSIR_K$ definition imply

$$\int_{x_1}^{x_4}\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx \ge 0 \text{ and hence } Q(\alpha_F)\ge 0. \qquad \text{Q.E.D.}$$

Lemmas 1 and 2 are required in the proof of Theorem.

**Theorem:** For a class of risk-averse decision-makers with $u''' \ge 0$, $\alpha_F \ge \alpha_G$ if

(a) $G$ $RSIR_K$ $F$

(b) $z_x \ge 0$, $z_{\alpha x} \ge 0$, $z_{\alpha\alpha} \le 0$ and $z_{xx} = 0$.

**Proof:** Let $k_1$, $k_2$, $k_3$ be the ordered values of $x^*$, *a*, *b*. With the points $m_1$, $m^0$ and $m_2$ used in Definition 2 where $x_2 \le m_1 \le m^0 \le m_2 \le x_4$, we consider the following four cases:

**Case (i)**: $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ when $x_1 \le x^* \le m_1$ (see Figure 2). We rewrite $Q(\alpha_F)$ in (5) as

$$\begin{aligned} Q(\alpha_F) &= u'(z(x_4,\alpha_F))\left[T_F(x_4,\alpha_F)-T_G(x_4,\alpha_F)\right] \\ &\quad -\int_{x_1}^{a} u''(z)\,z_x\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx \\ &\quad -\int_{k_2}^{x_4} u''(z)\,z_x\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx. \end{aligned}$$

Using these assumptions about $z(x,\alpha)$ and the $RSIR_K$ definition, the first term of $Q(\alpha_F)$ is non-negative. Therefore,

$$Q(\alpha_F) \geq -\int_{x_1}^{a} u''(z)\, z_x \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx \\ - \int_{a}^{x_4} u''(z)\, z_x \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx\,.$$

Consider the sign of the expression $\int_{x_1}^{a} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx$. First, if $\int_{x_1}^{a} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx \geq 0$, then Theorem follows from Lemma 2. Second, assuming that $\int_{x_1}^{a} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx \leq 0$, $Q(\alpha_F)$ becomes

$$Q(\alpha_F) \geq -\int_{x_1}^{a} u''(z)\, z_x \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx \\ - u''(z(b,\alpha_F))\, z_x \int_{a}^{x_4} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx\,.$$

Adding and subtracting $-u''(z(b,\alpha_F))\, z_x \int_{x_1}^{a} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx$ on the RHS of the above inequality gives

$$Q(\alpha_F) \geq -\int_{x_1}^{a} u''(z)\, z_x \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx \\ + u''(z(b,\alpha_F))\, z_x \int_{x_1}^{a} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx \\ - u''(z(b,\alpha_F))\, z_x \int_{x_1}^{x_4} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx\,.$$

From the RSIR$_K$ definition and $\int_{x_1}^{a} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx \leq 0$,

$$Q(\alpha_F) \geq -\int_{x_1}^{a} u''(z)\, z_x \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx\,, \tag{7}$$

where $-\int_{x_1}^{a} u''(z)\, z_x \left[T_F(x,\alpha_F) - T_G(x,\alpha_F)\right] dx$

$$= \int_{x_1}^{x_2} u''(z) z_x T_G(x_2, \alpha_F) dx - \int_{x_2}^{a} u''(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx .$$

Let $x^0$ be the value of $x$ that satisfies $z_\alpha(x, \alpha_F) = 0$ . $T_G(x_2, \alpha_F) = z_\alpha G(x_2) - \int_{x_1}^{x_2} z_{\alpha t} G(t) dt$ is non-positive since $z_\alpha \le 0$ for $x \in [x_1, x_2]$ . Therefore, (7) becomes

$$Q(\alpha_F) \ge -\int_{x_2}^{a} u''(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx . \tag{8}$$

Consider the term $[T_F(x, \alpha_F) - T_G(x, \alpha_F)]$ for $x \in [x_2, a)$.

$$\begin{aligned} &T_F(x, \alpha_F) - T_G(x, \alpha_F) \\ &\quad = z_\alpha [F(x) - G(x)] - \int_{x_2}^{x} z_{\alpha t} [F(t) - G(t)] dt \\ &\quad = z_\alpha \left[1 - \frac{1}{H_1(x)}\right] F(x) - \int_{x_2}^{x} z_{\alpha t} \left[1 - \frac{1}{H_1(t)}\right] F(t) dt \\ &\qquad \left(\text{Let } \frac{1}{H_1(x)} = h_1(x)\right) \\ &\quad = z_\alpha [1 - h_1(x)] F(x) - [1 - h_1(x)] \int_{x_2}^{x} z_{\alpha t} F(t) dt \\ &\qquad - \int_{x_2}^{x} h_1'(t) \int_{x_2}^{t} z_{\alpha k} F(k) dk dt . \end{aligned}$$

The last equality of the above equation can be obtained by integration by parts. Since, according to the condition (e) in Definition 2, $H_1(x)$ is non-decreasing and $H_1(x) \le 1$ for all $x \in [x_2, m_1)$ , $h_1(x)$ is non-increasing and $h_1(x) \ge 1$. Therefore,

$$\begin{aligned} &T_F(x, \alpha_F) - T_G(x, \alpha_F) \ge z_\alpha [1 - h_1(x)] F(x) - [1 - h_1(x)] \int_{x_2}^{x} z_{\alpha t} F(t) dt \\ &\ge [1 - h_1(x)] T_F(x, \alpha_F) . \end{aligned}$$

Thus (8) becomes

$$Q(\alpha_F) \geq -\int_{x_2}^{a} u''(z)\, z_x [1 - h_1(x)] T_F(x, \alpha_F)\, dx \geq$$
$$-\left[1 - h_1(x^c)\right] \int_{x_2}^{a} u''(z)\, z_x T_F(x, \alpha_F)\, dx \,.$$

Since $h_1(x)$ is non-increasing, $h_1(x^c) \geq 1$ and $\int_{x_2}^{a} u''(z)\, z_x T_F\ (x, \alpha_F)\, dx$ $\geq 0$ from Lemma 1 where $x^c$ is located on $(x_2, a)$ and $u''(z) z_x T_F$ $(x, \alpha_F)$ changes sign from positive to negative, $Q(\alpha_F) \geq 0$.

**Case (ii)**: $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m_1 \leq x^* \leq m_2$ (see Figures 3 and 4).

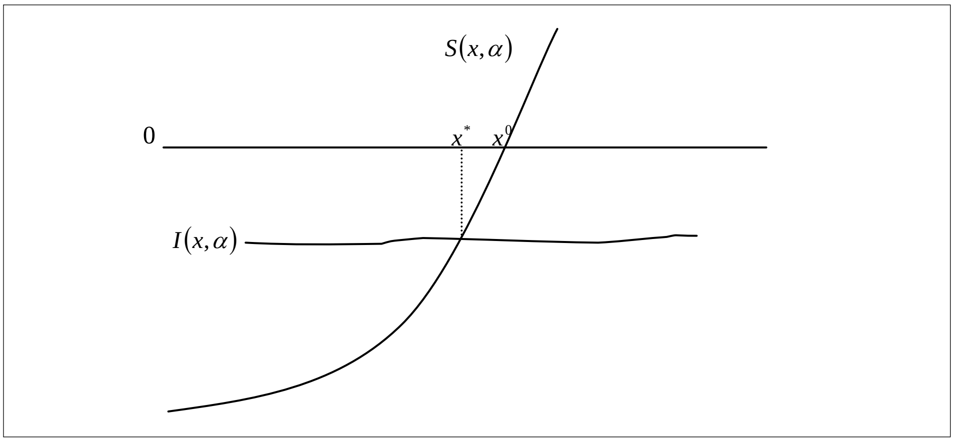
Before proving this case, we need to know the sign of $z_\alpha(x^*, \alpha_F) \int_{x_1}^{x^*} [F(x) - G(x)]\, dx$. Let $x^0$ be the value of $x$ satisfying $z_\alpha(x, \alpha_F) = 0$ and $m^0$ be the point satisfying $\int_{x_1}^{m^0} [F(x) - G(x)]\, dx = 0$. From the definition of RSIR$_K$ $\int_{x_1}^{x} [F(t) - G(t)]\, dt$ changes sign from negative to positive at the point $m^o$ on the interval $(m_1, m_2)$. We define that $\int_{x_1}^{x} F(t)\, dt = \hat{F}(x)$ and $\int_{x_1}^{x} G(t)\, dt = \hat{G}(x)$. Let $x^*$ denote the point where $[T_F(x, \alpha_F) - T_G(x, \alpha_F)]$ changes its sign when $z_\alpha$ has a sign change for each interval, that is, for $x^*$

$$z_\alpha [F(x) - G(x)] = \int_{x_1}^{x} z_{\alpha t} [F(t) - G(t)]\, dt \,. \tag{9}$$

We consider the following two sub-cases where $S(x, \alpha) =$ $z_\alpha [F(x) - G(x)]$ and $I(x, \alpha) = \int_{x_1}^{x} z_{\alpha t} [F(t) - G(t)]\, dt$.

**Case (ii-a)**: $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m_1 \leq x^* \leq m^0$ (see Figures 3 and 6).

**[Figure 6]** $m_1 \leq x^* \leq m^0$




In Figure 6, $z_\alpha\left(x^0, \alpha_F\right) = 0$ and $z_\alpha\left(x^*, \alpha_F\right) < 0$ since $\left[F\left(x^*\right) - G\left(x^*\right)\right]$ is positive from Definition 1 and $S\left(x^*, \alpha\right)$ is negative. From Lemma 2,

$$\begin{aligned}
&\int_{x_1}^{x^*} \left[T_F\left(x, \alpha_F\right) - T_G\left(x, \alpha_F\right)\right] dx \\
&\quad = \int_{x_1}^{x^*} z_\alpha \left[F(x) - G(x)\right] dx - \int_{x_1}^{x^*} \int_{x_1}^{x} z_{\alpha t}\left[F(t) - G(t)\right] dt dx \\
&\quad = z_\alpha\left(x^*, \alpha_F\right)\left[\hat{F}\left(x^*\right) - \hat{G}\left(x^*\right)\right] - \int_{x_1}^{x^*} z_{\alpha x}\left[\hat{F}(x) - \hat{G}(x)\right] dx \\
&\qquad - \int_{x_1}^{x^*} \int_{x_1}^{x} z_{\alpha t}\left[F(t) - G(t)\right] dt dx .
\end{aligned} \tag{10}$$

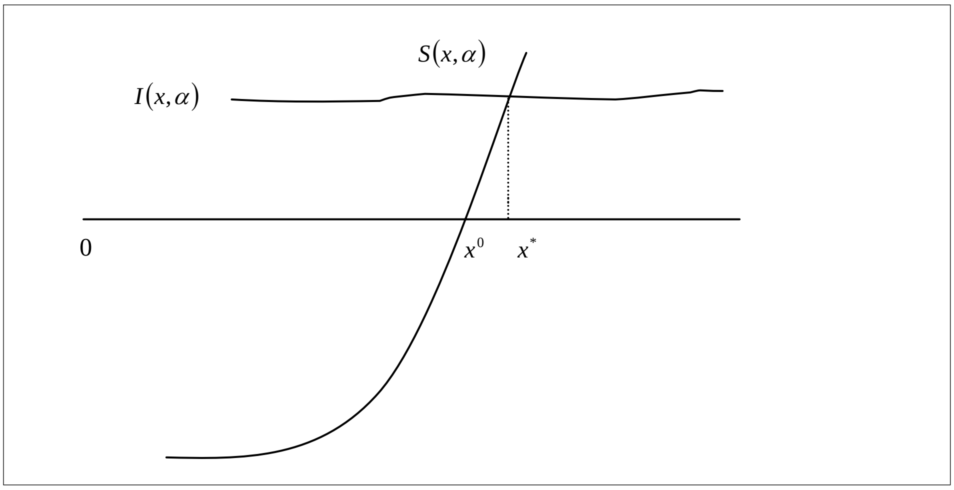
The RHS of (10) is non-negative since $z_\alpha\left(x^*, \alpha_F\right) < 0$, $\left[\hat{F}\left(x^*\right) - \hat{G}\left(x^*\right)\right]$ is non-positive and the condition (b) in Definition 1 is satisfied for the second and third terms in the RHS of (10). Therefore,

$$\int_{x_1}^{x^*} \left[T_F\left(x, \alpha_F\right) - T_G\left(x, \alpha_F\right)\right] dx \geq 0 .$$

This is sufficient for $Q\left(\alpha_F\right) \geq 0$.

**Case (ii-b)**: $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m^0 \leq x^* \leq m_2$ (see Figures 4 and 7).

**[Figure 7]** $m^0 \le x^* \le m_2$




In Figure 7, $z_\alpha\left(x^0,\alpha_F\right)=0$ and $z_\alpha\left(x^*,\alpha_F\right)>0$ since $\left[F\left(x^*\right)-G\left(x^*\right)\right]$ is positive from Definition 1 and $S\left(x^*,\alpha\right)$ is positive. The RHS of (10) is non-negative since $z_\alpha\left(x^*,\alpha_F\right)>0$ and $\left[\hat{F}\left(x^*\right)-\hat{G}\left(x^*\right)\right]$ is non-negative and the condition (b) in Definition 1 is satisfied. Therefore,

$$\int_{x_1}^{x^*}\left[T_F\left(x,\alpha_F\right)-T_G\left(x,\alpha_F\right)\right]dx \ge 0\,.$$

This is sufficient for $Q(\alpha_F) \ge 0$.

**Case (iii)**: $k_1=a$, $k_2=b$ and $k_3=x^*$ when $m_2 \le x^* \le x_4$ (see Figure 5). From the definition of $\alpha_F$

$$\int_{x_2}^{x_4} u'(z)\,z_\alpha\left(x,\alpha_F\right)dF(x)=0\,.$$

Integrating by parts gives

$$\begin{aligned}0 &= \int_{x_2}^{x_4} u'(z)\,z_\alpha dF(x) = u'\left(z\left(x_4,\alpha_F\right)\right)T_F\left(x_4,\alpha_F\right)\\ &\quad -\int_{x_2}^{x_4} u''(z)\,z_x T_F\left(x,\alpha_F\right)dx \le u'\left(z\left(x_4,\alpha_F\right)\right)T_F\left(x_4,\alpha_F\right)\end{aligned}$$

$$-u''\left(z\left(x^c,\alpha_F\right)\right)z_x\int_{x_2}^{x_4}T_F\left(x,\alpha_F\right)dx\,.$$

The above inequality holds since $T_F\left(x,\alpha_F\right)$ changes sign from negative to positive at the point $x^c$ and $-\int_{x_2}^{x_4}u''(z)\,z_xT_F\left(x,\alpha_F\right)dx \le -u''\left(z\left(x^c,\alpha_F\right)\right)z_x\int_{x_2}^{x_4}T_F\left(x,\alpha_F\right)dx$ because $-u''(z)\,z_x$ is non-negative and decreasing in $x$. It implies $\int_{x_2}^{x_4}T_F\left(x,\alpha_F\right)dx \ge 0$ , but this implies $\int_s^{x_4}T_F\left(x,\alpha_F\right)dx \ge 0$ for all $s\in\left[x_2,x_4\right]$. In the case of $s=b$ , integrations of $\int_b^{x_4}\left[T_F\left(x,\alpha_F\right)-T_G\left(x,\alpha_F\right)\right]dx$ by parts gives

$$\begin{aligned}&\int_b^{x_4}\left[T_F\left(x,\alpha_F\right)-T_G\left(x,\alpha_F\right)\right]dx\\&\qquad=\int_b^{x_4}\left\{z_\alpha\left[F(x)-G(x)\right]-\int_{x_1}^{x}z_{\alpha t}\left[F(t)-G(t)\right]dt\right\}dx\end{aligned}\tag{11}$$

Consider the second term with minus sign in the bracket on the RHS of (11).

$$\begin{aligned}&-\int_{x_1}^{x}z_{\alpha t}\left[F(t)-G(t)\right]dt=\\&\qquad-\int_{x_1}^{b}z_{\alpha t}\left[F(t)-G(t)\right]dt-\int_b^{x}z_{\alpha t}\left[F(t)-G(t)\right]dt\,.\end{aligned}\tag{12}$$

The first term with minus sign on the RHS of (12) is non-positive since $b$ is located at the right side of $m^0$, where $m^0$ is the point satisfying $\int_{x_1}^{m^0}\left[F(x)-G(x)\right]dx=0$ . Therefore,

$$-\int_{x_1}^{x}z_{\alpha t}\left[F(t)-G(t)\right]dt \le -\int_b^{x}z_{\alpha t}\left[1-\frac{1}{H_2(t)}\right]F(t)\,dt$$

$$\left[Let\,\frac{1}{H_2(t)}=h_2(t)\right]$$

$$\leq -[1-h_2(x)]\int_b^x z_{\alpha t}F(t)\,dt - \int_b^x h_2'(t)\int_b^t z_{\alpha k}F(k)\,dk dt$$
$$\leq -[1-h_2(x)]\int_b^x z_{\alpha t}F(t)\,dt \,. \tag{13}$$

The last inequality in (13) can be obtained by integration by parts. Since, according to the condition (f) in Definition 2, $H_2(x)$ is non-increasing and $H_2(x)\leq 1$ for all $x\in(m_2,x_3]$, $h_2(x)$ is non-decreasing and $h_2(x)\geq 1$ for all $x\in(m_2,x_3]$.

Therefore, adding and subtracting $-[1-h_2(x)]\int_{x_1}^b z_{\alpha t}F(t)\,dt$ on the RHS of (13) gives

$$-\int_{x_1}^x z_{\alpha t}[F(t)-G(t)]\,dt \leq -[1-h_2(x)]\int_{x_1}^x z_{\alpha t}F(t)\,dt$$
$$+[1-h_2(x)]\int_{x_1}^b z_{\alpha t}F(t)\,dt \leq -[1-h_2(x)]\int_{x_1}^x z_{\alpha t}F(t)\,dt \,. \tag{14}$$

Since $[1-h_2(x)]$ is non-positive and $\int_{x_1}^b z_{\alpha t}F(t)\,dt$ is non-negative, the last inequality in (14) holds. Thus,

$$T_F(x,\alpha_F)-T_G(x,\alpha_F)\leq z_\alpha[1-h_2(x)]F(x)$$
$$-[1-h_2(x)]\int_{x_1}^x z_{\alpha t}F(t)\,dt = [1-h_2(x)]T_F(x,\alpha_F)\,. \tag{15}$$

Therefore (11) becomes

$$\int_b^{x_4}[T_F(x,\alpha_F)-T_G(x,\alpha_F)]\,dx \leq \int_b^{x_4}[1-h_2(x)]T_F(x,\alpha_F)\,dx$$
$$\leq [1-h_2(x^c)]\int_b^{x_4}T_F(x,\alpha_F)\,dx \leq 0\,. \tag{16}$$

The second inequality in (16) holds since $h_2(x)$ is non-decreasing and $T_F(x,\alpha_F)$ changes sign from negative to positive at $x^c$. Thus, the left-hand-side of (16) is non-positive because $[1-h_2(x^c)]$ is non-positive and $\int_b^{x_4}T_F(x,\alpha_F)\,dx$ is non-negative where $T_F(x,\alpha_F)$ changes sign from negative to positive and $\int_{x_2}^{x_4} T_F(x,\alpha_F)\,dx \geq 0$ . Since integrating of $\int_{x_1}^{x_4}\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]$ by parts gives $\int_{x_1}^{x_4}\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]$ $dx \geq 0$ and $\int_{b}^{x_4}\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx \leq 0$ is non-positive from (16), it implies $\int_{x_1}^{k_2(=b)}\left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx \geq 0$ from Lemma 2. Therefore, $Q(\alpha_F) \geq 0$ . Q.E.D.

Compared with the comparative statics result derived by Black and Bulkely, Theorem includes a larger set of changes in distribution but shows somewhat stronger restriction on the structure of the decision model. When the structural restriction $z_{xx} = 0$ is added, the relationship between the results in Black and Bulkley's analysis and in Theorem shows a trade-off between the restrictions on the set of decision-makers and the set of changes in distribution. The concept of prudence is used in Eeckhoudt and Kimball (1992) and Eeckhoudt et al. (1996) who analyzed the impact of an increase in background risk on the choice variable.

When we assume that the payoff function is linear in the choice variable, the form of $z(x,\alpha)$ may be expressed as $z(x,\alpha)=\alpha(x-c)+z_0$ , where $z_0$ and c are exogenous constants. This linear payoff prevails in many economic applications such as the standard portfolio model, the optimal behavior of a competitive firm with constant marginal costs, behavior of cooperative firm, the problem of hiring workers, the coinsurance problem and others.

Note that the risk preferences of the decision makers for a relatively strong increase in risk in the K-L-L-S sense' ($\text{RSIR}_\text{K}$) and a simple increases in risk across $r$ in the K-L-L-S sense (sIR( $r$ )) in Ryu and Kim (2005) are same, but the distributional changes for them are different. When the payoff function is linear, $\hat{F}(x)$ and $\hat{G}(x)$ must be intersected at the point *r* for the sIR( $r$ ) order but the point *r* can be located in any place in the interval $(x_2, x_3)$ for the $\text{RSIR}_\text{K}$ order.

## IV. CONCLUSIONS

This paper proposes a new notion of the subset of K-L-L-S increases in risk called a 'relatively strong increase in risk in the K-L-L-S sense' ($RSIR_K$). We show that, by restricting the payoff function to be linear in the random variable ( $z_{xx} = 0$ ) and limiting our analysis to decision-makers who are downside risk-averse ($u''' \geq 0$), we are able to generate comparative statics results for a relatively strong increases in risk in the K-L-L-S sense including the set of Rothschild-Stiglitz increases in risk. This implies that K-L-L-S increases in risk extend the Rothschild-Stiglitz definition of risk to a larger set of cumulative distribution functions, but use somewhat stronger restrictions on the structure of the decision model and the set of decision-makers.

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