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Evolutionarily Stable Correlation*

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Most results of evolutionary games are restricted only to the Nash equilibrium. In this paper, we introduce an analogue of the evolutionarily stable strategy (ESS) for correlated equilibria. We introduce a new concept—the evolutionarily stable correlation (ESC)—and prove that it generalizes the ESS. We also study analogues of perfection and properness for correlated equilibria and discuss their relationships with the ESC.

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I. Introduction

The Nash equilibrium is a central concept in the theory of normal form games, which captures a situation in which players choose their strategies independently. Aumann (1974) extended the Nash equilibrium to the notion of the correlated equilibrium, in which players base their strategy choices on their observations of correlating signals.[1] There have been a number of refinements made to the Nash equilibrium, e.g., the evolutionary stable strategy (ESS), perfect equilibrium, and proper equilibrium. The main purpose of this paper is to examine an analogue of

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[1] As pointed out in Hart and Mas-Colell (2000), p.1128, lines 3-4, "it is hard to exclude a priori the possibility that correlating signals are amply available to the players..."

the ESS for the correlated equilibrium, which we denote as the evolutionarily stable correlation (ESC). We also examine analogues of perfection and properness for correlated equilibria and investigate their relationships with the ESC.

The concept of the ESS is based on the evolutionary selection process, which is based on the idea that individuals who poorly adapt to a given environment will eventually disappear by natural selection. Speaking more rigorously, the ESS is a conventional (i.e., monomorphic) strategy adopted in a population that cannot be invaded by a small group of mutants. We adapt this evolutionary idea to correlated equilibria. We consider a large population model with a uniform random matching process and we assume that a given (conventional) random device recommends actions to matched players. When players adopt the "obedient" strategy, the device generates a probability distribution of the joint actions played—the conventional correlation. We investigate the stability of this correlation in relation to the mutations of players' strategies when assigning actions actually played that are conditional on the recommended actions. Suppose a group of mutants appears in the population, and all of them use the same action assignment strategy, which differs from the obedient strategy. Although the (conventional) random device remains the same, the resulting correlation of the joint actions played differs from the conventional correlation. We state that the conventional correlation is an ESC when an incumbent using the obedient strategy performs better than a mutant using the nonobedient strategy.

This description of the ESC is based on the direct mechanism, in which a random device directly recommends actions. Our formal definition of the ESC also covers the case of an indirect mechanism, in which signals of the device need not be actions. We prove that the formulation of an ESC using the direct mechanism is equivalent to that using an indirect mechanism (Proposition 2).

The ESS refines the Nash equilibrium and deals with evolutionary stability under the restriction of independent plays. Similarly, the ESC refines the correlated equilibrium and deals with evolutionary stability without the restriction of independent plays.[2] We show that an ESC is a correlated equilibrium, but not vice versa. We characterize the ESC with best-reply conditions (Proposition 1) and prove that the ESC is a generalization of the ESS (Proposition 5) in the same manner as the correlated equilibrium is a generalization of the Nash equilibrium. We also characterize the ESC in relation to local superiority (Proposition 7).

We introduce the notion of a perfect and proper correlated equilibrium and show that an ESC is a proper correlated equilibrium and that a proper correlated equilibrium is a perfect correlated equilibrium (Propositions 9, 11 and 12).

[2] What we mean by a refinement is that the ESS is contained in the set of Nash equilibria and the ESC is contained in the set of correlated equilibria. A formal refinement often requires the existence of such refinement, but neither the ESS nor the ESC satisfies this requirement.

However, none of the converse statements are true.

II. Evolutionarily Stable Correlation

2.1. Correlated Equilibrium and Mechanism

Consider a two-player symmetric finite normal form game $G=\{S_1,S_2;u_1,u_2\}$, where $S_1=S_2=S$, and S is a nonempty finite set, with the payoff function $u:S\times S\to\mathbb{R}$ such that $u_1(s_1,s_2)=u(s_1,s_2)=u_2(s_2,s_1)$ for all $(s_1,s_2)\in S\times S$. A *(pure) action* is an element $s\in S$, and a *mixed action* (or *mixed strategy*) is an $\zeta\in\Delta(S)$.[3] A *correlation* is a $\zeta\in\Delta(S\times S)$. A correlation ζ is *symmetric* if $\zeta(s_1,s_2)=\zeta(s_2,s_1)$ for all $(s_1,s_2)\in S\times S$. Aumann (1974, 1987) extended the notion of the Nash equilibrium to allow players' correlated actions.

Definition 1 (cf. Aumann (1974, 1987)) A symmetric $\zeta\in\Delta(S\times S)$ is a *correlated equilibrium* if for all $s_1\in S$,

$$\sum_{s_2\in S}u(s_1,s_2)\zeta(s_1,s_2)\geq\sum_{s_2\in S}u(s_1',s_2)\zeta(s_1,s_2)\quad\text{for all } s_1'\in S. \tag{1}$$

A correlated equilibrium can be viewed as the outcome of a Bayesian maximization of players in relation to a random device, which is a lottery mechanism for selecting a private message for each agent.

In formal terms, a *random device* is a tuple $F=(M,\pi)$, where M is a nonempty finite set, the message space is $M_1\times M_2$ with $M_1=M_2=M$, and π is a symmetric correlation over the message space.

For any random device $F=(M,\pi)$, an *assignment function* is a function $\delta:M\to\Delta(S)$, i.e., δ assigns a mixed action $\delta(m)$ for every observed message m. For each $m\in M$ and $s\in S$, we use $\delta(s\mid m)$ to denote the probability that s is played under mixed action $\delta(m)$. We use Q^M to denote the set of assignment functions, i.e., $Q^M=\{\delta\mid\delta:M\to\Delta(S)\}$.[4]

An assignment function δ is *pure* if for all $m\in M$ there exists an $s\in S$ such that $\delta(s\mid m)=1$. We use T^M to denote the set of pure assignment functions.

An *assignment pair* is a pair $(\delta_1,\delta_2)\in Q^M\times Q^M$. An assignment pair (δ_1,δ_2) is *symmetric* if $\delta_1=\delta_2$.

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[3] For any finite set X, we use $\Delta(X)$ to denote the set of probability measures over X.

[4] We view Q^M as the set $\Pi_{m\in M}\Delta(S)$, which is a subset in the linear space $\mathbb{R}^{\#(S)\#(M)}$ endowed with the Euclidean topology. For every $\delta,\delta'\in Q^M$ and every $\lambda\in[0,1],\lambda\delta+(1-\lambda)\delta'$ is the element in Q^M where $(\lambda\delta+(1-\lambda)\delta')(s\mid m)=\lambda\delta(s\mid m)+(1-\lambda)\delta'(s\mid m)$ for all $s\in S$ and all $m\in M$. Thus, Q^M is a compact and convex set.

We say that a correlation $\mu \in \Delta(S \times S)$ is *generated by* a random device $F = (M, \pi)$ and an assignment pair $(\delta_1, \delta_2) \in Q^M \times Q^M$, and we write $\mu = K^{M,\pi}(\delta_1, \delta_2)$, if for all $(s_1, s_2) \in S \times S$,

$$\mu(s_1, s_2) = \sum_{(m_1, m_2) \in M \times M} \pi(m_1, m_2) \delta_1(s_1 \mid m_1) \delta_2(s_2 \mid m_2) . \tag{2}$$

Given any random device $F = (M, \pi)$ and any assignment pair $(\delta_1, \delta_2) \in Q^M \times Q^M$, the expected payoff to agent 1 is as follows:

$$U^{M,\pi}(\delta_1, \delta_2) = \sum_{(s_1, s_2) \in S \times S} u(s_1, s_2) \mu(s_1, s_2), \quad \text{where} \quad \mu = K^{M,\pi}(\delta_1, \delta_2) . \tag{3}$$

By symmetry, the expected payoff for agent 2 is $U^{M,\pi}(\delta_2, \delta_1)$.

For any random device $F = (M, \pi)$, we define the symmetric normal form game $\Gamma^{M,\pi} = \{R_1, R_2; U_1, U_2\}$ where the strategy sets $R_1 = R_2 = Q^M$, and the payoff functions $U_1(\delta_1, \delta_2) = U_2(\delta_2, \delta_1) = U^{M,\pi}(\delta_1, \delta_2)$ for all $(\delta_1, \delta_2) \in Q^M \times Q^M$.

We frequently consider a device in which $M = S$. To simplify the notation, for a device with $F = (S, \pi)$ we often drop the superscript " S " in the expressions " $K^{S,\pi}$," " $U^{S,\pi}$," " Q^S ," " T^S ," and " $\Gamma^{S,\pi}$," and write: " K^π ," " U^π ," " Q ," " T ," and " Γ^π ."

For any random device $F = (M, \pi)$, a *Nash equilibrium* in $\Gamma^{M,\pi}$ is a pair $(\delta_1, \delta_2) \in Q^M \times Q^M$ such that for all $i, j \in \{1, 2\}$ with $i \neq j$:

$$U^{M,\pi}(\delta_i, \delta_j) \geq U^{M,\pi}(\delta_i', \delta_j) \quad \text{for all} \quad \delta_i' \in Q^M . \tag{4}$$

A *mechanism* is a tuple $\mathcal{M} = (M, \pi, \delta)$ in which (M, π) is a random device, and $\delta \in Q^M$.

For any symmetric correlation $\zeta \in \Delta(S \times S)$, we are particularly interested in the *direct mechanism* for ζ , i.e., the mechanism $\mathcal{M} = (S, \zeta, \delta^{id})$ where $\delta^{id} \in Q$ is the *obedient* (identity) assignment function. That is, for all $s \in S$:

$$\delta^{id}(s \mid s) = 1 \quad \text{and} \quad \delta^{id}(s' \mid s) = 0 \quad \text{for} \quad s' \in S \setminus \{s\} . \tag{5}$$

Under the direct mechanism $\mathcal{M} = (S, \zeta, \delta^{id})$, the direct random device $F = (S, \zeta)$ generates messages (s_1, s_2) . When an agent i receives a message $s_i \in S$, it means that the device recommends that he plays the action s_i .

For every symmetric correlation $\zeta \in \Delta(S \times S)$ and any mechanism $\mathcal{M} = (M, \pi, \delta)$, we say that ζ is *realized* by $\mathcal{M}$ if

a) $F = (M, \pi)$ and (δ, δ) generate ζ , i.e., $K^{M,\pi}(\delta, \delta) = \zeta$,

b) (δ,δ) is a Nash equilibrium in $\Gamma^{M,\pi}$. (6)

In view of (1), it is clear that a symmetric $\zeta \in \Delta(S\times S)$ is a correlated equilibrium if and only if the obedient assignment pair $(\delta^{id},\delta^{id})$ is a Nash equilibrium in Γ^{ζ}. That is, being obedient (playing the recommended actions) is the best response for an agent if the opponent is also obedient.

In general, it is well known that a symmetric $\zeta \in \Delta(S\times S)$ is a correlated equilibrium if and only if there is a (not necessarily direct) mechanism realizing ζ.[5]

2.2. Evolutionary Stability for Correlation and Mechanism

To study the evolutionary stability of a correlated equilibrium, we study the evolutionary stability of a mechanism that realizes a correlated equilibrium.

First, we review the standard concept of evolutionary stability for a Nash equilibrium. For any $(x,y)\in\Delta(S)\times\Delta(S)$, we use $V(x,y)$ to denote the (expected) payoff for the first player when (x,y) is used, i.e.,

$$V(x,y)=\sum_{(s_1,s_2)\in S\times S} x(s_1)y(s_2)u(s_1,s_2)\,. \tag{7}$$

Definition 2 (Maynard Smith (1982)) For a given symmetric game G, an *evolutionarily stable strategy (ESS)* is an $x\in\Delta(S)$ such that for every $y\in\Delta(S)$ with $x\neq y$, there is an $\bar{\varepsilon}>0$ such that for all $\varepsilon\in(0,\bar{\varepsilon})$:

$$(1-\varepsilon)V(x,x)+\varepsilon V(x,y)>(1-\varepsilon)V(y,x)+\varepsilon V(y,y)\,. \tag{8}$$

Intuitively, an ESS x is a conventional (monomorphic) mixed action played by a population that cannot be invaded by a small group of mutations playing another mixed action y.

It is well known that x is an ESS if and only if x satisfies the following best-reply conditions:

a) $V(y,x)\le V(x,x)$ for all $y\in\Delta(S)$;
b) $V(y,x)=V(x,x)\Rightarrow V(y,y)<V(x,y)$ for all $y\in\Delta(S)$ with $y\neq x$. (9)

(Cf. Weibull (1995), p.37, Proposition 2.1.)

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[5] Indeed, ζ is realized by the direct mechanism (S,ζ,δ^{id}) if and only if ζ is realized by some (not necessarily direct) mechanism (M,π,δ). The "only if" part is clear; the "if" part can be proved by the argument as used in the third and fourth paragraphs in the proof of Proposition 2.

To study the evolutionary stability of a mechanism, we must take note of a special feature. Although we always treat any distinct mixed actions y, y' as different when defining an ESS, it is not always natural to treat any distinct assignments δ, δ' as different for a mechanism. They may generate the same correlation over $S \times S$ no matter what assignment functions the opponents use. We formalize this intuition as follows:

Definition 3 Consider any random device $F = (M, \pi)$. For any assignment functions $\delta, \delta' \in Q^M$, we say that δ and δ' are *equivalent* and write $\delta \simeq^{M,\pi} \delta'$ if

$$K^{M,\pi}(\delta, \delta'') = K^{M,\pi}(\delta', \delta'') \quad \text{for all } \delta'' \in Q^M . \tag{10}$$

To simplify the notation, when $M = S$, we often write "$\simeq^{\pi}$" instead of "$\simeq^{S,\pi}$."

It follows easily from Definition 3 that for any random device $F = (M, \pi)$ and any $\delta \in Q^M$, the set $\{\delta' \in Q^M : \delta' \simeq^{M,\pi} \delta\}$ is closed and convex.

We provide an example of a pair of equivalent assignments and a pair of nonequivalent assignments in Examples 1 and 2, respectively.

The following property is immediate based on the definition of equivalence.

Lemma 1 *Consider any random device* $F = (M, \pi)$, *and any assignment functions* $\delta, \delta' \in Q^M$. *If* $\delta \simeq^{M,\pi} \delta'$, *then*

$$K^{M,\pi}(\delta, \delta) = K^{M,\pi}(\delta', \delta') = K^{M,\pi}(\delta, \delta') = K^{M,\pi}(\delta', \delta) . \tag{11}$$

Proof: Let $\delta \simeq^{M,\pi} \delta'$. Substitute $\delta'' = \delta$ into (10). We have: $K^{M,\pi}(\delta, \delta) = K^{M,\pi}(\delta', \delta)$. As the correlation $K^{M,\pi}(\delta, \delta)$ is symmetric, so is $K^{M,\pi}(\delta', \delta)$. Hence, $K^{M,\pi}(\delta, \delta) = K^{M,\pi}(\delta', \delta) = K^{M,\pi}(\delta, \delta')$. By substituting $\delta'' = \delta'$ into (10), we also have the following: $K^{M,\pi}(\delta', \delta') = K^{M,\pi}(\delta, \delta')$. Thus, (11) holds. Q.E.D.

Condition (11) has the following implications for the expected payoffs:

$$U^{M,\pi}(\delta, \delta) = U^{M,\pi}(\delta', \delta') = U^{M,\pi}(\delta, \delta') = U^{M,\pi}(\delta', \delta) . \tag{12}$$

The converse of Lemma 1 is true when $M = S$ and $\delta = \delta^{id}$, as shown by the following lemma.

Lemma 2 *Consider a random device* $F = (S, \pi)$. *For all* $\delta' \in Q$,

$$K^{\pi}(\delta', \delta^{id}) = \pi \quad \Rightarrow \quad \delta' \simeq^{\pi} \delta^{id} . \tag{13}$$

Proof: Consider any $\delta' \in Q$ with $K^{\pi}(\delta', \delta^{id}) = \pi$. Then, for all $(s_1, s_2) \in S \times S$, we see that

$$\begin{aligned} \pi(s_1, s_2) &= \sum_{(s_1', s_2') \in S \times S} \pi(s_1', s_2') \delta'(s_1 \mid s_2') \delta^{id}(s_2 \mid s_2') \\ &= \sum_{s_1' \in S} \pi(s_1', s_2) \delta'(s_1 \mid s_1') \ . \end{aligned} \tag{14}$$

Now, consider any $\delta'' \in Q$. We denote $\mu_1 = K^{\zeta}(\delta', \delta'')$ and $\mu_2 = K^{\zeta}(\delta^{id}, \delta'')$. Then, for all $(s_1, s_2) \in S \times S$, we have the following:

$$\begin{aligned} \mu_1(s_1, s_2) &= \sum_{(s_1', s_2') \in S \times S} \pi(s_1', s_2') \delta'(s_1 \mid s_2') \delta''(s_2 \mid s_2') \\ &= \sum_{s_2' \in S} \delta''(s_2 \mid s_2') \sum_{s_1' \in S} \pi(s_1', s_2') \delta'(s_1 \mid s_1') \\ &= \sum_{s_2' \in S} \delta''(s_2 \mid s_2') \pi(s_1, s_2') \\ &= \sum_{(s_1', s_2') \in S \times S} \delta''(s_2 \mid s_2') \pi(s_1', s_2') \delta^{id}(s_1 \mid s_1') \\ &= \mu_2(s_1, s_2) . \end{aligned} \tag{15}$$

Thus, $K^{\pi}(\delta^{id}, \delta'') = K^{\pi}(\delta', \delta'')$. Q.E.D.

Lemma 2 is useful for checking the $\simeq^{M,\pi}$ relation for the case in which $M = S$ and $\delta = \delta^{id}$.

Example 1 Let $S = \{X, Y\}$ and device $F = (S, \pi)$, where $\pi = (1/4)XX + (1/4)XY + (1/4)YX + (1/4)YY$. Let $\delta' \in Q$ be such that $\delta'(X) = (1/2)X + (1/2)Y$, and $\delta'(Y) = (1/2)X + (1/2)Y$. Then, $K^{\pi}(\delta', \delta^{id}) = \pi$. Thus, based on Lemma 2, $\delta' \simeq^{\pi} \delta^{id}$.

The following example shows that for an arbitrary random device $F = (M, \zeta)$, condition (11) does not imply condition (10).

Example 2 Let $S = \{X, Y\}$ and device $F = (M, \pi)$, where $M = \{a, b, c, d\}$ and $\pi = (1/4)aa + (1/4)bb + (1/4)cd + (1/4)dc$. Consider the assignment functions δ, δ', $\delta'' \in Q^M$, where

$$\begin{aligned} &\delta(a) = X \quad && \delta'(a) = Y \quad && \delta''(a) = X \\ &\delta(b) = Y && \delta'(b) = X && \delta''(b) = X \\ &\delta(c) = X && \delta'(c) = Y && \delta''(c) = X \end{aligned}$$

$$\delta(d) = Y \qquad \delta'(d) = X \qquad \delta''(d) = Y. \tag{16}$$

Then, we have $K^{M,\pi}(\delta,\delta) = K^{M,\pi}(\delta',\delta') = K^{M,\pi}(\delta,\delta') = K^{M,\pi}(\delta',\delta) = (1/4)XX$ $+(1/4)XY + (1/4)YX + (1/4)YY$. However, $K^{M,\pi}(\delta,\delta'') = (1/4)XX$ $+(1/2)YX +$ $(1/4)XY \neq (1/2)XX + (1/4)YX + (1/4)YY = K^{M,\pi}(\delta',\delta'')$.

With the equivalence relation $\simeq^{M,\pi}$, we define the following evolutionary stability for an assignment in the game $\Gamma^{M,\pi}$.

Definition 4 For any random device $F = (M,\pi)$, an $*$*-evolutionarily stable strategy* (ESS^*) for the game $\Gamma^{M,\pi}$ is a $\delta \in Q^M$ such that for every $\delta' \in Q^M$ with $\delta' \not\simeq^{M,\pi} \delta$, there is an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\bar{\varepsilon})$:

$$(1-\varepsilon)U^{M,\pi}(\delta,\delta) + \varepsilon U^{M,\pi}(\delta,\delta') > (1-\varepsilon)U^{M,\pi}(\delta',\delta) + \varepsilon U^{M,\pi}(\delta',\delta'). \tag{17}$$

(Note that for those δ' with $\delta' \simeq^{M,\pi} \delta$, based on (12), the ">" in (17) cannot hold, as we must instead have the "=.")

We obtain a similar characterization for (9a,b) for our ESS^*.

Proposition 1 *For any random device* $F = (M,\pi)$, *a* $\delta \in Q^M$ *is an* ESS^* *for* $\Gamma^{M,\pi}$ *if and only if it satisfies the following best-reply conditions:*

a) $U^{M,\pi}(\delta',\delta) \leq U^{M,\pi}(\delta,\delta)$ *for all* $\delta' \in Q^M$;

b) $U^{M,\pi}(\delta',\delta) = U^{M,\pi}(\delta,\delta) \Rightarrow U^{M,\pi}(\delta',\delta') < U^{M,\pi}(\delta,\delta')$
for all $\delta' \in Q^M$ *with* $\delta' \not\simeq^{M,\pi} \delta$. (18)

Proof: This can be proved by standard ESS arguments (cf. Weibull (1995), pp. 36-37). For the reader's convenience, we provide the details here.

Consider any correlation π, and consider the random device $F = (M,\pi)$.

First, let δ be an ESS^* for $\Gamma^{M,\pi}$. To see (18a), suppose there exists a $\delta' \in Q^M$ such that $U^{M,\pi}(\delta',\delta) > U^{M,\pi}(\delta,\delta)$. Then, $\delta' \not\simeq^{M,\pi} \delta$. Furthermore, for all small $\varepsilon > 0$, we have the following:

$$(1-\varepsilon)U^{M,\pi}(\delta,\delta) + \varepsilon U^{M,\pi}(\delta,\delta') < (1-\varepsilon)U^{M,\pi}(\delta',\delta) + \varepsilon U^{M,\pi}(\delta',\delta'). \tag{19}$$

Thus, δ is not an ESS^*.

For (18b), suppose $U^{M,\pi}(\delta',\delta) = U^{M,\pi}(\delta,\delta)$ and $U^{M,\pi}(\delta',\delta') > U^{M,\pi}(\delta,\delta')$ for some $\delta' \in Q^M$ with $\delta' \not\simeq^{M,\pi} \delta$. Then, (19) holds for all $\varepsilon > 0$. Hence, δ is not an ESS^*.

Conversely, let δ satisfy (18). Consider any $\delta' \in Q^M$ with $\delta' \neq^{M,\pi} \delta$. Based on (18a), either $U^{M,\pi}(\delta',\delta) < U^{M,\pi}(\delta,\delta)$ or $U^{M,\pi}(\delta',\delta) = U^{M,\pi}(\delta,\delta)$.

(Case 1) Suppose $U^{M,\pi}(\delta',\delta) < U^{M,\pi}(\delta,\delta)$. Then, (17) holds for all small $\varepsilon > 0$.

(Case 2) Suppose $U^{M,\pi}(\delta',\delta) = U^{M,\pi}(\delta,\delta)$. Then, based on (18b), we have the following: $U^{M,\pi}(\delta',\delta') < U^{M,\pi}(\delta,\delta')$. As such, (17) holds for all $\varepsilon > 0$.

Thus, δ is an ESS^*. Q.E.D.

Using the ESS^* Definition 4, we now define evolutionary stability for a correlation.

Definition 5 *An evolutionarily stable correlation (ESC) is a symmetric* $\zeta \in \Delta(S \times S)$*, such that there exists a mechanism* $\mathcal{M} = (M,\pi,\delta)$ *that evolutionarily realizes* ζ*, i.e.,*

a) $F = (M,\pi)$ and (δ,δ) generate ζ, i.e., $K^{M,\pi}(\delta,\delta) = \zeta$,

b) δ is an ESS^* for $\Gamma^{M,\pi}$. (20)

Definition 5 uses an arbitrary mechanism. In the following, Definition 6 uses only the direction mechanism $\mathcal{M} = (S,\zeta,\delta^{id})$ for ζ. As shown in Proposition 2, the two definitions are equivalent.

Definition 6 *An evolutionarily stable correlation (ESC)* is a symmetric $\zeta \in \Delta(S \times S)$ such that δ^{id} is an ESS^* for Γ^{ζ}.

In the sense of Definition 6, a symmetric $\zeta \in \Delta(S \times S)$ is an ESC if the following applies: for every assignment function $\delta' \in Q$ with $\delta' \neq^{\zeta} \delta^{id}$, there is an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\bar{\varepsilon})$:

$$(1-\varepsilon)U^{\zeta}(\delta^{id},\delta^{id}) + \varepsilon U^{\zeta}(\delta^{id},\delta') > (1-\varepsilon)U^{\zeta}(\delta',\delta^{id}) + \varepsilon U^{\zeta}(\delta',\delta'). \tag{21}$$

Equivalently, if:

a) $U^{\zeta}(\delta',\delta^{id}) \leq U^{\zeta}(\delta^{id},\delta^{id})$ for all $\delta' \in Q$;

b) $U^{\zeta}(\delta',\delta^{id}) = U^{\zeta}(\delta^{id},\delta^{id}) \Rightarrow U^{\zeta}(\delta',\delta') < U^{\zeta}(\delta^{id},\delta')$ for all $\delta' \in Q$ with $\delta' \neq^{\zeta} \delta^{id}$. (22)

Next, we discuss the intuition of an ESC ζ in view of Definition 6 and (21). Consider a large population that obeys the signal sent by the conventional random device $F = (S,\zeta)$. Now, suppose that a small group of mutants appears. Mutants do not obey the signal sent by the conventional random device. Pairs of individuals in this bimorphic population are repeatedly drawn to play the game with a uniform matching probability. In playing the game, the pair of players faces the conventional random device $F=(S,\zeta)$. The correlation ζ is evolutionarily stable if the incumbents with obedient strategies perform better than the mutants whose strategies are not equivalent to the obedient strategy. Thus, the evolutionary force drives out the mutants.

The intuition for the ESC given in Definition 5 is similar.

The ESC is the analogue of the ESS for correlated equilibria.

We end this section by noting the equivalence between Definitions 5 and 6.

Proposition 2 *Let* $\zeta \in \Delta(S\times S)$ *be symmetric. Then,* ζ *is an ESC in the sense of Definition 5 if and only if it is an ESC in the sense of Definition 6.*

Proof: Definition 6 implies Definition 5, a fortiori. We need to only prove the converse.

Let ζ be an ESC in the sense of Definition 5. Let mechanism $\mathcal{M}=(M,\pi,\delta)$ evolutionarily realize ζ , i.e., (20) holds. Now, consider the random device $F=(S,\zeta)$ and the obedient assignment $\delta^{id}\in Q$. We must show that δ^{id} is an ESS^* for Γ^{ζ} , equivalently, (22) holds.

As F and (δ,δ) generate ζ (based on (20a)), we have the following:

$$K^{M,\pi}(\delta,\delta)=\zeta=K^{\zeta}(\delta^{id},\delta^{id})\,. \tag{23}$$

As such,

$$U^{M,\pi}(\delta,\delta)=U^{\zeta}(\delta^{id},\delta^{id})\,. \tag{24}$$

Now, consider any $\delta'\in Q$. Define $\tilde{\delta}'\in Q^{M}$ as the composition of δ' and δ , i.e.,

$$\tilde{\delta}'(s\,|\,m)=\sum_{s'\in S}\delta'(s\,|\,s')\delta(s'\,|\,m) \qquad \text{for all } s\in S \text{ and } m\in M\,. \tag{25}$$

Then,

$$\begin{aligned} K^{M,\pi}(\tilde{\delta}',\delta)&=K^{\zeta}(\delta',\delta^{id})\,,\\ K^{M,\pi}(\tilde{\delta}',\tilde{\delta}')&=K^{\zeta}(\delta',\delta')\,. \end{aligned} \tag{26}$$

As such,

$$U^{M,\pi}(\tilde{\delta}',\delta)=U^{\zeta}(\delta',\delta^{id})$$
$$U^{M,\pi}(\delta,\tilde{\delta}')=U^{\zeta}(\delta^{id},\delta')$$
$$U^{M,\pi}(\tilde{\delta}',\tilde{\delta}')=U^{\zeta}(\delta',\delta'). \quad (27)$$

To prove (22a), consider any $\delta'\in Q$. Then,

$$\begin{aligned} U^{\zeta}(\delta',\delta^{id}) &= U^{M,\pi}(\tilde{\delta}',\delta) \quad \text{(by (27))} \\ &\le U^{M,\pi}(\delta,\delta) \quad \text{(by (18a))} \\ &= U^{\zeta}(\delta^{id},\delta^{id}) \quad \text{(by (24))}. \end{aligned} \quad (28)$$

Thus, (22a) holds.

To prove (22b), we suppose that it does not hold. That is, there exists an assignment $\delta'\in Q$ with $\delta'\not\simeq^{\zeta}\delta^{id}$ such that

$$U^{\zeta}(\delta',\delta^{id})=U^{\zeta}(\delta^{id},\delta^{id}) \quad \text{and} \quad U^{\zeta}(\delta',\delta')\ge U^{\zeta}(\delta^{id},\delta'). \quad (29)$$

Based on (24) and (27), we have the following:

$$U^{M,\pi}(\tilde{\delta}',\delta)=U^{M,\pi}(\delta,\delta) \quad \text{and} \quad U^{M,\pi}(\tilde{\delta}',\tilde{\delta}')\ge U^{M,\pi}(\delta,\tilde{\delta}'). \quad (30)$$

We now show that $\tilde{\delta}'\not\simeq^{M,\pi}\delta$. Suppose this is not the case, i.e., $\tilde{\delta}'\simeq^{M,\pi}\delta$. Then,

$$\begin{aligned} K^{\zeta}(\delta',\delta^{id}) &= K^{M,\pi}(\tilde{\delta}',\delta) \quad \text{(by (26))} \\ &= K^{M,\pi}(\delta,\delta) \quad (\text{as } \tilde{\delta}'\simeq^{M,\pi}\delta) \\ &= K^{\zeta}(\delta^{id},\delta^{id}) \quad \text{(by (23))}. \end{aligned} \quad (31)$$

Therefore, based on Lemma 2, we have $\delta'\simeq^{\zeta}\delta^{id}$, and a contradiction is derived. As $\tilde{\delta}'\not\simeq^{M,\pi}\delta$, (30) contradicts (18b). This establishes (22b). Q.E.D.

2.3. Properties for ESC

In this section, we study the properties of the ESC. Our analysis relies on Definition 6, which uses the direct mechanism. We use the best-reply characterization (22).

Proposition 3 immediately follows from (22a).

Proposition 3 *If* $\zeta\in\Delta(S\times S)$ *is an ESC, then* ζ *is a correlated equilibrium.*

A symmetric $\zeta \in \Delta(S \times S)$ is a *strict correlated equilibrium* if for all $s_1 \in S$ with $\sum_{s_2 \in S} \zeta(s_1, s_2) > 0$,

$$\sum_{s_2 \in S} u(s_1, s_2)\zeta(s_1, s_2) > \sum_{s_2 \in S} u(s_1', s_2)\zeta(s_1, s_2) \text{ for all } s_1' \in S \text{ with } s_1' \neq s_1 . \tag{32}$$

Proposition 4 immediately follows from Proposition 1 (with (22)) and Lemma 2.[6]

Proposition 4 *A strict correlated equilibrium* ζ *is an ESC.*

The following example shows that the converse of Proposition 3 is not true, i.e., not every correlated equilibrium is an ESC.

Example 3 Consider the following game :

$$\begin{array}{ccc} & X & Y \\ X & (1,1) & (0,0) \\ Y & (0,0) & (1,1) \end{array} . \tag{33}$$

It is easy to verify that a symmetric $\zeta \in \Delta(S \times S)$ is a correlated equilibrium if and only if it has the form $\zeta = aXX + bYY + cXY + cYX$ with $a \geq c$ and $b \geq c$.

We suppose that $c = 0$, then, ζ is a strict correlated equilibrium. As such, based on Proposition 4, ζ is an ESC.

Suppose $c > 0$. We show that ζ is an ESC if and only if $a > c$ and $b > c$. First, suppose $a, b > c$. In this case, ζ is a strict correlated equilibrium; therefore, it is an ESC. Next, suppose $a = c$ or $b = c$. In this case, ζ is not an ESC. Without loss of generality, let $a = c$. Consider the assignment δ where $\delta(X) = Y$ and $\delta(Y) = Y$. Then, $U^{\zeta}(\delta, \delta^{id}) = c + b = a + b = U^{\zeta}(\delta^{id}, \delta^{id})$, but $U^{\zeta}(\delta, \delta) = 1 > b + c = U^{\zeta}(\delta^{id}, \delta)$. Thus, ζ violates (22b) and is not an ESC.

Thus, not every correlated equilibrium is an ESC.

The following example shows that the converse of Proposition 4 is not true, i.e., an ESC is not necessarily a strict correlated equilibrium.

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[6] To prove Proposition 4, consider any $\delta' \in Q$. There are two cases:

(Case 1) Suppose $\delta'(s_1 | s_1) \neq 1$ for some $s_1 \in S$ with $\sum_{s_2 \in S} \zeta(s_1, s_2) > 0$. As ζ is a strict correlated equilibrium, we have $U^{\zeta}(\delta^{id}, \delta^{id}) - U^{\zeta}(\delta', \delta^{id}) = \sum_{s_1 \in S}[\sum_{s_2 \in S} u(s_1, s_2)\zeta(s_1, s_2) - \sum_{s_2 \in S} (\sum_{s_1' \in S} u(s_1', s_2)\delta'(s_1' | s_1))\zeta(s_1, s_2)] > 0$.

(Case 2) Suppose $\delta'(s_1 | s_1) = 1$ for all $s_1 \in S$ with $\sum_{s_2 \in S} \zeta(s_1, s_2) > 0$. Then, $K^{\zeta}(\delta', \delta^{id}) = \zeta$. Based on Lemma 2, we have $\delta' \simeq^{\zeta} \delta^{id}$.

These two cases establish (22).

Example 4 Consider the following game in which $S = \{X, Y\}$ and the payoffs are as follows:

$$\begin{array}{ccc} & X & Y \\ X & (1,1) & (1,1) \\ Y & (1,1) & (0,0) \end{array} . \tag{34}$$

Then, the pure action X is an ESS. Therefore, $\zeta = XX$ is an ESC by Proposition 5 below. This ESC is clearly not a strict correlated equilibrium.

In the following, Proposition 5 shows the relationship between the ESS and the ESC.

For any $x, y \in \Delta(S)$, we denote $x \times y$ as their product, i.e., $x \times y \in \Delta(S \times S)$, where

$$(x \times y)(s_1, s_2) = x(s_1) y(s_2) \quad \text{for all } (s_1, s_2) \in S \times S . \tag{35}$$

Proposition 5 *Let* $x \in \Delta(S)$ *and* $\zeta = x \times x$ *. Then,* x *is an ESS if and only if* ζ *is an ESC.*

Proof: ("Only if" Part) Let x be an ESS. Based on (9a), (x, x) is a Nash equilibrium. As such, ζ is a correlated equilibrium. Therefore, ζ satisfies (22a). Thus, (22b) remains to be proved. Consider any $\delta' \in Q$ with $\delta' \neq^{\zeta} \delta^{id}$. We choose the strategy $y \in \Delta(S)$, where

$$y(s) = \sum_{s' \in S} x(s') \delta'(s \mid s') \quad \text{for all } s \in S . \tag{36}$$

Then, based on the definition of K^{ζ}, we have the following:

$$\begin{aligned} &K^{\zeta}(\delta^{id}, \delta^{id}) = x \times x \\ &K^{\zeta}(\delta^{id}, \delta') = x \times y \\ &K^{\zeta}(\delta', \delta^{id}) = y \times x \\ &K^{\zeta}(\delta', \delta') = y \times y . \end{aligned} \tag{37}$$

Hence,

$$\begin{aligned} &U^{\zeta}(\delta^{id}, \delta^{id}) = V(x, x) \qquad U^{\zeta}(\delta', \delta^{id}) = V(y, x) \\ &U^{\zeta}(\delta^{id}, \delta') = V(x, y) \qquad U^{\zeta}(\delta', \delta') = V(y, y) . \end{aligned} \tag{38}$$

As $\delta' \neq^{\zeta} \delta^{id}$, we must have $y \neq x$. Then, (22b) follows from (9b).
("If" Part) Let ζ be an ESC. For any $y \in \Delta(S)$ with $y \neq x$, we choose the assignment $\delta' \in Q$, where

$$\delta'(s' \mid s) = y(s') \qquad \forall s, s' \in S. \tag{39}$$

Then, (37) and (38) follow. As $y \neq x$, based on (37), we have $\delta^{id} \neq^{\zeta} \delta'$. Therefore, (9a) and (9b) follow from (22a) and (22b), respectively. Q.E.D.

Proposition 5 shows that the ESC is a generalization of the ESS.

It is well known that for any symmetric game there are at most a finitely many ESS's. A standard proof for the finiteness property follows from the best-reply characterization (9) (cf. Weibull (1995), p. 41.). In particular, the property (9) has a strong implication for the support Supp$(\cdot)$ of ESS, including that Supp$(y) \not\subseteq$ Supp(x) for every ESS x and y with $x \neq y$. This support property ensures that the set of ESSs is finite. However, the best-reply characterization (22) does not imply such a restriction for the support of an ESC.[7] In (9), the ESS x is a choice variable; however, in (22), the ESC ζ is not. Without such a support restriction, the set of ESCs need not be finite. In Example 3, there are only two ESSs but infinitely many ESCs. In fact, using (21), it is easy to verify that if the set of ESCs is nonempty, then it is convex. Thus, if there is more than one ESC, then there are infinitely many.

The ESS and ESC do share common characteristics. For example, it is well known that the existence of an ESS is not guaranteed. In the next example, we show that the same is true for the ESC.

Example 5 Consider the following rock-scissors-paper game (cf. Weibull (1995), p. 28, Example 1.12):

$$\begin{array}{cccc} & X & Y & Z \\ X & (1,1) & (2,0) & (0,2) \\ Y & (0,2) & (1,1) & (2,0) \\ Z & (2,0) & (0,2) & (1,1) \end{array}. \tag{40}$$

If $\zeta \in \Delta(S \times S)$ is symmetric, then it takes the form $aXX + bYY + cZZ + dXY + dYX + eXZ + eZX + fYZ + fZY$. It is easily verified that if ζ is a correlated equilibrium, then we must have $a = b = c = d = e = f = 1/9$, which is the product

[7] That is, in general, it is not true that Supp$(\zeta) \not\subseteq$ Supp(ζ') for every ESC ζ and ζ' with $\zeta \neq \zeta'$.

measure of the symmetric Nash equilibrium with the mixed strategy $(1/3)X+(1/3)Y+(1/3)Z$. This Nash equilibrium mixed strategy is not an ESS, as any mutation of a pure action will perform equally well as that of a mixed action. Consequently, this game has no ESC.[8]

Another feature shared by the ESS and ESC is local superiority. Any ESS earns a higher payoff against all nearby mutants than the mutants earn against themselves. In formal terms, an $x \in \Delta(S)$ is *locally superior* if

$$\text{there is a neighborhood } N \text{ of } x \text{ in } \Delta(S) \text{ such that } V(x,y) > V(y,y) \text{ for all } y \in N \text{ with } y \neq x. \tag{41}$$

It is well known that an $x \in \Delta(S)$ is an ESS if and only if is locally superior. (Cf. Weibull (1995), p. 45, Proposition 2.6.)

We develop an analogue for the ESC. For simplicity, we do it only for the direct mechanism.

Definition 7 Given a random device $F=(S,\zeta)$, an assignment δ is *locally superior* if there is a neighborhood N of δ in Q such that $U^{\zeta}(\delta,\delta') > U^{\zeta}(\delta',\delta')$ for all $\delta' \in N$ with $\delta' \neq^{\zeta} \delta$. We say ζ is *locally superior* if the obedient assignment δ^{id} is locally superior.

Proposition 6 shows that Definition 7 is a generalization of the standard definition of local superiority for the ESS.

Proposition 6 *Let $x \in \Delta(S)$ and $\zeta = x \times x$. Then, ζ is locally superior (in the sense of Definition 7) if and only if x is locally superior (in the sense of (41)).*

Proof: It is well known that an $x \in \Delta(S)$ is an ESS if and only if it is locally superior in the standard ESS sense (as defined in (41)). Based on Propositions 5 and 7, x is an ESS if and only if $\zeta = x \times x$ is an ESC if and only if it is locally superior in the sense of Definition 7. Q.E.D.

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[8] The following is an example of a case in which there exist some correlated equilibria that are not Nash equilibria, but no ESC exists. Consider the following:

	X	Y
X	(0,0)	(0,0) .
Y	(0,0)	(0,0)

For all $a \in (0,1)$, the correlation $aXX+(1-a)YY$ is a correlated equilibrium but not a Nash equilibrium; however, no ESC exists.

Proposition 7 is an analogue of similar ESS results (cf. Weibull (1995), Proposition 2.6). Our proof requires Lemma 3, which shows that the invasion barrier $\overline{\varepsilon}$ can be taken uniformly for all mutants.

Lemma 3 *Let symmetric* $\zeta \in \Delta(S \times S)$ *be an ESC. Then,*
(a) There exists a uniform invasion barrier $\overline{\varepsilon} > 0$*, i.e., for all* $\delta' \in Q$ *with* $\delta' \neq^{\zeta} \delta^{id}$*, (21) holds for all* $\varepsilon \in (0, \overline{\varepsilon})$.
(b) ζ *is locally superior.*

Proof: See the Appendix.

Proposition 7 *A symmetric* $\zeta \in \Delta(S \times S)$ *is an ESC if and only if* ζ *is locally superior.*

Proof: The "only if" part is given by Lemma 3b. We need to only prove the "if" part.
Suppose an open set $N \subseteq Q$ is a neighborhood of δ^{id} as given in Definition 7. Note that for every $\delta' \in Q$, there is a small $\overline{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon})$, $w = \varepsilon\delta' + (1-\varepsilon)\delta^{id} \in N$. If $\delta' \neq^{\zeta} \delta^{id}$, then we have $w \neq^{\zeta} \delta^{id}$. Based on the local superiority of δ^{id}, we have the following:

$$U^{\zeta}(\delta^{id}, w) > U^{\zeta}(w, w). \tag{42}$$

As U^{ζ} is bilinear, we have the following: $U^{\zeta}(w,w) = \varepsilon U^{\zeta}(\delta', w) + (1-\varepsilon) U^{\zeta}(\delta^{id}, w)$. Thus, (42) ensures that $U^{\zeta}(\delta^{id}, w) > U^{\zeta}(\delta', w)$, i.e., $(1-\varepsilon)U^{\zeta}(\delta^{id}, \delta^{id}) + \varepsilon U^{\zeta}(\delta^{id}, \delta') > (1-\varepsilon)U^{\zeta}(\delta', \delta^{id}) + \varepsilon U^{\zeta}(\delta', \delta')$. Therefore, ζ is an ESC. Q.E.D.

To conclude this section, we compare our work with related work of Shmida and Peleg (1997) and Cripps (1991). Consider a given symmetric game G with S and u. Their work shows that a symmetric $\zeta \in \Delta(S \times S)$ is a strict correlated equilibrium if and only if ζ is generated by some pure strategy δ that is an ESS in some asymmetric animal conflict (cf. Selten (1980))[9] with role asymmetry and payoff-irrelevant roles. (Cf. Shmida and Peleg (1997), Theorem 4.1, and Cripps (1991), Theorem.) Such an asymmetric conflict can be identified as a game $\Gamma^{M,\pi}$, where (M, π) is a random device such that the symmetric $\pi \in \Delta(M \times M)$ satisfies the role asymmetry property (i.e., $\pi(m,m) = 0$ for all $m \in M$), and the message set M is interpreted as set of possible types.

[9] The term asymmetric animal conflict is from the study by Selten (1980) and Shmida and Peleg (1997) and is called a *simple contest* in the study by Cripps (1991). Shmida and Peleg (1997) restricted their study to symmetric bimatrix games, and Cripps (1991) focused on general bimatrix games.

Our work shares some similarities with that of the preceding authors in that we commonly relate the notion of correlated equilibria to the idea of evolutionary stability by using phenotypic conditional behavior with random devices (M,π). However, two distinctions should be made clear.

First, in formalizing conditional behavior, these authors required a role assignment measure π to satisfy the role asymmetry property. We do not require any such role asymmetry, although we allow it.

Second, the ESC here is more general than these authors' idea of realizing a strict correlated equilibrium. While our ESC generalizes the ESS, these authors failed to realize all ESSs as their strict correlated equilibria failed to include all of the ESSs in a given normal form game.[10]

III. Evolutionary Stability, Perfection, and Properness

It is well known that for an ESS x, the symmetric pair (x,x) is a perfect and proper equilibrium. Here we provide similar results for the ESC. For this purpose, we need analogues of perfect and proper equilibria in our correlation context. Our approach is restricted to the direct random device $F=(S,\zeta)$.

First, we recall the definitions of a perfect and proper equilibrium in the given game G.

We require the following notation:

$$\overset{\circ}{\Delta}(S)=\{x\in\Delta:(\forall s\in S)(x(s)>0\},$$
$$P=\{\varepsilon=(\varepsilon_s)_{s\in S}:[(\forall s\in S)(\varepsilon_s>0)]\,\&\,(\sum_{s\in S}\varepsilon_s<1)\},$$
$$\Delta_\varepsilon(S)=\{x\in X:(\forall s\in S)(x(s)\geq\varepsilon_s)\}\qquad\forall\varepsilon\in P. \tag{43}$$

That is, $\overset{\circ}{\Delta}(S)$ is the set of completely mixed action, P is the set of perturbations, and $\Delta_\varepsilon(S)$ is the set of mixed actions allowable for the given perturbation ε.

Definition 8 A *perfect equilibrium* in G is a pair $(x_1,x_2)\in\Delta(S)\times\Delta(S)$ that

[10] For example, consider the following Hawk-Dove game:

	H	D
H	(0,0)	(4,2).
D	(2,4)	(2,2)

The mixed strategy $(1/2)H+(1/2)D$ is an ESS. However, its product measure is neither a strict Nash equilibrium nor a strict correlated equilibrium.

satisfies one of the following equivalent conditions:[11]

Condition A There exists a sequence $(x_1^n, x_2^n) \in \mathring{\Delta}(S) \times \mathring{\Delta}(S)$ such that $(x_1^n, x_2^n) \to (x_1, x_2)$, and for each n and each $i, j \in \{1,2\}$ with $i \neq j$, x_i is a best reply to x_j^n over $\Delta(S)$, i.e.,

$$V(x_i, x_j^n) \geq V(x', x_j^n) \qquad \forall x' \in \Delta(S). \tag{44}$$

Condition B There exists a sequence $(\varepsilon_1^n, \varepsilon_2^n) \in P \times P$ and a sequence $(x_1^n, x_2^n) \in \Delta_{\varepsilon_1^n}(S) \times \Delta_{\varepsilon_2^n}(S)$ such that $(\varepsilon_1^n, \varepsilon_2^n) \to (0,0)$, $(x_1^n, x_2^n) \to (x_1, x_2)$, and for each n and each $i, j \in \{1,2\}$ with $i \neq j, x_i^n$ is a best reply to x_j^n over $\Delta_{\varepsilon_i^n}(S)$, i.e.:

$$V(x_i^n, x_j^n) \geq V(x', x_j^n) \qquad \forall x' \in \Delta_{\varepsilon_i^n}(S). \tag{45}$$

Definition 9 A *proper equilibrium* in G is a pair $(x_1, x_2) \in \Delta(S) \times \Delta(S)$ that satisfies the following:

Condition C There exists a sequence $\varepsilon^n \in (0,1)$ and a sequence $(x_1^n, x_2^n) \in \mathring{\Delta}(S) \times \mathring{\Delta}(S)$ such that $\varepsilon^n \to 0$, $(x_1^n, x_2^n) \to (x_1, x_2)$, and for each n and each $i, j \in (1,2\}$ with $i \neq j$:

$$V(s', x_j^n) < V(s'', x_j^n) \Rightarrow x_i^n(s') \leq \varepsilon^n x_i^n(s'') \qquad \forall s', s'' \in S. \tag{46}$$

We now extend the definition of a perfect Nash equilibrium to correlated equilibria. We require the following notation:

$$\begin{aligned} &\mathring{Q}(S) = \{\delta \in Q : (\forall s, s' \in S)(\delta(s' \mid s) > 0)\}, \\ &P^S = \{\varepsilon = (\varepsilon_{s,s'})_{s,s' \in S} : [(\forall s, s' \in S)(\varepsilon_{s,s'} > 0)] \,\&[(\forall s \in S)(\sum_{s' \in S} \varepsilon_{s,s'} < 1)]\}, \\ &Q_\varepsilon = \{\delta \in Q : (\forall s, s' \in S)(\delta(s' \mid s) \geq \varepsilon_{s,s'})\} \quad \text{for all } \varepsilon \in P^S. \end{aligned} \tag{47}$$

Each $\delta \in \mathring{Q}$ assigns a completely mixed action $\delta(s)$ for all $s \in S$. Each $\varepsilon \in P^S$ is a perturbation for strategies in Q, and Q_ε is the set of allowable assignments under the perturbation ε. For any $\delta \in Q_\varepsilon$, δ assigns a completely mixed action $\delta(s) \in \Delta_{\varepsilon_s}(S)$ for all $s \in S$, where $\varepsilon_s = (\varepsilon_{s,s'})_{s' \in S} \in P$.

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[11] Cf. Mas-Colell et al. (1995), Proposition 8.F.1.

Definition 10 A *perfect correlated equilibrium* is a symmetric $\zeta \in \Delta(S \times S)$ such that there is a $\delta \in D = \{\delta' \in Q : \delta' \simeq \delta^{id}\}$ that satisfies one of the following equivalent conditions:

Condition A′ There exists a sequence $(\delta_1^n, \delta_2^n) \in \mathring{Q} \times \mathring{Q}$ such that $(\delta_1^n, \delta_2^n) \to (\delta, \delta)$, and for each n and each $j \in \{1,2\}$, δ is a best reply to δ_j^n over Q, i.e.,

$$U^{\zeta}(\delta, \delta_j^n) \geq U^{\zeta}(\delta', \delta_j^n) \quad \forall \delta' \in Q. \tag{48}$$

Condition B′ There exists a sequence $(\varepsilon_1^n, \varepsilon_2^n) \in P^S \times P^S$ and a sequence $(\delta_1^n, \delta_2^n) \in Q_{\varepsilon_1^n} \times Q_{\varepsilon_2^n}$ such that $(\varepsilon_1^n, \varepsilon_2^n) \to (0,0)$, $(\delta_1^n, \delta_2^n) \to (\delta, \delta)$, and for each n and each $i, j \in \{1,2\}$ with $i \neq j$, δ_i^n is a best reply to δ_j^n over $Q_{\varepsilon_i^n}$, i.e.,

$$U^{\zeta}(\delta_i^n, \delta_j^n) \geq U^{\zeta}(\delta', \delta_j^n) \quad \forall \delta' \in Q_{\varepsilon_i^n}. \tag{49}$$

In the Appendix, we provide a proof for the equivalence between Conditions A′ and B′.

To define a proper correlated equilibrium, we require the following notation. For every $s, s' \in S$, and $\delta \in Q$, we define the following:

$$U^{\zeta}(s', \delta \mid s) = \sum_{s_2 \in S} \sum_{s_2' \in S} \zeta(s, s_2) \delta(s_2' \mid s_2) u(s', s_2'). \tag{50}$$

Conditional on the event that player 2 uses the assignment δ, and player 1 is recommended to play action s, $U^{\zeta}(s', \delta \mid s)$ can be viewed as the "expected payoff" earned by player 1 by using action s'. (More precisely, the true conditional expected payoff is $(\sum_{s_2 \in S} \zeta(s, s_2))^{-1} U^{\zeta}(s', \delta \mid s)$ for $(\sum_{s_2 \in S} \zeta(s, s_2) > 0)$.[12]

In our definition of a proper correlated equilibrium, for each $s \in S$, the term $U^{\zeta}(s', \delta \mid s)$ plays a role similar to that of $V(s', y)$ in (46).

The terms $U^{\zeta}(\cdot, \cdot \mid \cdot)$ are also useful for finding the best replies. Consider any $(\delta_1, \delta_2) \in Q \times Q$. Note the following:

$$U^{\zeta}(\delta_1, \delta_2) = \sum_{s_1 \in S} \sum_{s_1' \in S} \delta_1(s_1' \mid s_1) U^{\zeta}(s_1', \delta_2 \mid s_1). \tag{51}$$

Furthermore, for all $s \in S$:

[12] Our "conditional expected" payoff notation $U^{\zeta}(\cdot, \cdot \mid \cdot)$ resembles the commonly used symbol $\Pr(\cdot \mid \cdot)$ for denoting a conditional probability measure.

$$\text{if } \sum_{s_2 \in S} \zeta(s, s_2) = 0, \text{ then } U^{\zeta}(s', \delta \mid s) = 0 \quad \text{for all } s' \in S. \tag{52}$$

Then, we have the following:

Lemma 4 *For every* $(\delta_1, \delta_2) \in Q \times Q$ *, the following two conditions are equivalent:*
(a) δ_1 *is a best reply to* δ_2 *over* Q *, i.e.,* $U^{\zeta}(\delta_1, \delta_2) \geq U^{\zeta}(\delta', \delta_2)$ *for all* $\delta' \in Q$ *.*
(b) For every $s \in S$ *with* $\sum_{s_2 \in S} \zeta(s, s_2) > 0$ *:*

$$\delta_1(s' \mid s) > 0 \Rightarrow U^{\zeta}(s', \delta_2 \mid s) = \max_{s'' \in S} U^{\zeta}(s'', \delta_2 \mid s) \qquad \forall s' \in S. \tag{53}$$

Definition 11 A *proper correlated equilibrium* is a symmetric $\zeta \in \Delta(S \times S)$ such that there is a $\delta \in D = \{\delta' \in Q : \delta' \simeq \delta^{id}\}$ that satisfies the following:

Condition C′ There exists a sequence of $\varepsilon^n \in (0,1)$ and a sequence $(\delta_1^n, \delta_2^n) \in \mathring{Q} \times \mathring{Q}$ such that $\varepsilon^n \to 0$, $(\delta_1^n, \delta_2^n) \to (\delta, \delta)$, and for every n , every $i, j \in \{1,2\}$ with $i \neq j$ and every $s \in S$:

$$U^{\zeta}(s', \delta_j^n \mid s) < U^{\zeta}(s'', \delta_j^n \mid s) \Rightarrow \delta_i^n(s' \mid s) \leq \varepsilon^n \delta_i^n(s'' \mid s) \qquad \forall s', s'' \in S. \tag{54}$$

The following proposition shows that Definitions 10 and 11 are generalizations of perfect and proper Nash equilibria.

Proposition 8 *Let* $x \in \Delta(S)$ *and* $\zeta = x \times x \in \Delta(S \times S)$*. Then,*
(a) (x, x) *is a perfect equilibrium if and only if* ζ *is a perfect correlated equilibrium.*
(b) (x, x) *is a proper equilibrium if and only if* ζ *is a proper correlated equilibrium.*

Proof: See the Appendix.

Here we sketch the main idea of the proof of Proposition 8. We use two functions. First, for each $y \in \Delta(S)$, we define $\delta_y \in Q$ by

$$\delta_y(s' \mid s) = y(s') \qquad \forall s, s' \in S. \tag{55}$$

Second, for each $\delta \in Q$, we define $y_\delta \in \Delta(S)$ by

$$y_\delta(s) = \sum_{s' \in S} x(s') \delta(s \mid s') \qquad \forall s \in S. \tag{56}$$

Then, based on the definition of K^{ζ}, we have

a) $K^{\zeta}(\delta_y, \delta_{\tilde{y}}) = y \times \tilde{y}$ for all $y, \tilde{y} \in \Delta(S)$.

b) $K^{\zeta}(\delta, \tilde{\delta}) = y_{\delta} \times y_{\tilde{\delta}}$ for all $\delta, \tilde{\delta} \in Q$. (57)

Using these two functions, we can transform a sequence (x_1^n, x_2^n) as given in Condition A into a sequence (δ_1^n, δ_2^n) as given in Condition A′, and vice versa. Part (a) of the proposition follows, and part (b) can be proved similarly.

The relationship between a perfect correlated equilibrium and a proper correlated equilibrium is the same as that for the Nash equilibrium.

Proposition 9 *(a) A proper correlated equilibrium is a perfect correlated equilibrium. (b) The converse of (a) is not true.*

Proof: (Part a) It suffices to consider any $\delta \in Q$, and prove that Condition C′ implies Condition A′. First, consider any sequences ε^n and (δ_1^n, δ_2^n), as given in Condition C′. We claim that

$$\delta \text{ is a best reply to } \delta_2^n \text{ over } Q \text{ for all large } n. \quad (58)$$

As $\delta_1^n \to \delta$ and $\varepsilon^n \to 0$, we can choose an $\varepsilon \in (0,1)$ such that for all large n:

a) $\varepsilon^n \le \varepsilon$

b) for all $s \in S$ with $\sum_{s_2 \in S} \zeta(s, s_2) > 0$:

$$\delta(s' \mid s) > 0 \Rightarrow \delta_1^n(s' \mid s) > \varepsilon \quad \text{for all } s' \in S. \quad (59)$$

Consider any large n and any $s, s' \in S$ with $\sum_{s_2 \in S} \zeta(s, s_2) > 0$ and $\delta(s' \mid s) > 0$. Then,

$$\delta_1^n(s' \mid s) > \varepsilon \ge \varepsilon^n \ge \varepsilon^n \delta_1^n(s'' \mid s) \quad \forall s'' \in S \quad (60)$$

Hence, based on (54),

$$U^{\zeta}(s', \delta_2^n \mid s) \ge U^{\zeta}(s'', \delta_2^n \mid s) \quad \forall s'' \in S. \quad (61)$$

That is, $U^{\zeta}(s', \delta_2^n \mid s) = \max_{s'' \in S} U^{\zeta}(s'', \delta_2^n \mid s)$. Based on Lemma 4, we see that (58) follows. Similarly, δ is a best reply to δ_1^n over Q for all large n.

Thus, by taking away the initial terms if necessary, we can assume that the sequence (δ_n^1, δ_n^2) satisfies (48) for all n. Hence, the sequence satisfies the property given in Condition A′.

(Part b) When $\zeta \in \Delta(S \times S) = x \times x$, the notion of a proper equilibrium coincides with that of proper correlated equilibrium. It is well known that a perfect Nash equilibrium is not necessarily a proper Nash equilibrium, and this is also true for a symmetric equilibrium in a symmetric game. (For example, see Myerson (1978), p. 78, last paragraph.) Q.E.D.

Definitions 10 and 11 use the set Q. We provide two other natural formalizations by using the set T of pure assignments. We utilize a natural linear function mapping $\Delta(T)$ onto Q. In particular, we choose the linear function $h : \Delta(T) \to Q$ defined by

$$(h(x))(s' \mid s) = \sum_{\hat{\delta} \in \{\hat{\delta} \in T \mid \hat{\delta}(s) = s'\}} x(\hat{\delta}) \quad \text{for all } s, s' \in S. \tag{62}$$

As shown in Lemma 5 in the Appendix, the function h maps $\Delta(T)$ onto Q.[13] It also maps $\mathring{\Delta}(T)$ onto $\mathring{Q}$.

For every $x, y \in \Delta(T)$, we define

$$K^{\zeta}(x,y) = \sum_{(\delta_1,\delta_2) \in T \times T} x(\delta_1) y(\delta_2) K^{\zeta}(\delta_1,\delta_2),$$
$$U^{\zeta}(x,y) = \sum_{(\delta_1,\delta_2) \in T \times T} x(\delta_1) y(\delta_2) U^{\zeta}(\delta_1,\delta_2). \tag{63}$$

Then, based on (62) we have

$$K^{\zeta}(x,y) = K^{\zeta}(h(x),h(y)),$$
$$U^{\zeta}(x,y) = U^{\zeta}(h(x),h(y)). \tag{64}$$

For a given $\zeta \in \Delta(S \times S)$, we define the symmetric finite normal form game $\tilde{\Gamma}^{\zeta} = \{\tilde{R}_1, \tilde{R}_2, U_1, U_2\}$, where the strategy sets are $\tilde{R}_1 = \tilde{R}_2 = T$, and for all $x, y \in \Delta(T)$, the payoffs $U_1(x,y) = U_2(y,x) = U^{\zeta}(x,y)$.

[13] However, in general h is not a one-to-one mapping. For example, let $S = \{X, Y\}$, and $x, y \in \Delta(T)$, where

$$x = (1/2)\delta^1 + (1/2)\delta^2, \quad y = (1/2)\delta^3 + (1/2)\delta^4,$$

and

$$\begin{array}{llll} \delta^1(X) = X & \delta^2(X) = Y & \delta^3(X) = X & \delta^4(X) = Y \\ \delta^1(Y) = Y & \delta^2(Y) = X & \delta^3(Y) = X & \delta^4(Y) = Y. \end{array}$$

Then, $h(x) = h(y)$.

Definition 12 A *perfect correlated equilibrium* is a symmetric $\zeta \in \Delta(S \times S)$ such that there is an $x \in h^{-1}(D)$ such that (x,x) is a perfect equilibrium in the game $\tilde{\Gamma}^{\zeta}$, where $D = \{\delta \in Q : \delta \simeq^{\zeta} \delta^{id}\}$.

We now explain the meaning of the requirement "$x \in h^{-1}(D)$." For any $\delta \in Q$, $\delta \in D$ means that $\delta \simeq^{\zeta} \delta^{id}$, i.e., they are equivalent in the sense that $K^{\zeta}(\delta,\delta') = K^{\zeta}(\delta^{id},\delta')$ for all $\delta' \in Q$. For any $x \in \Delta(T)$, $x \in h^{-1}(D)$ means that $h(x) \simeq^{\zeta} \delta^{id}$. As such, $K^{\zeta}(h(x),\delta) = K^{\zeta}(\delta^{id},\delta)$ for all $\delta \in Q$. As h maps $\Delta(T)$ onto Q, based on (64), we have the following:

$$K^{\zeta}(x,y) = K^{\zeta}(\delta^{id},y) \quad \text{for all } y \in \Delta(T). \tag{65}$$

Thus, x and δ^{id} are "equivalent."

Proposition 10

a) Definitions 10 and 12 are equivalent.

b) Definition 12 is equivalent to the following apparent strengthening of Definition 12:

A symmetric $\zeta \in \Delta(S \times S)$ is a perfect correlated equilibrium if $(\delta^{id},\delta^{id})$ is a perfect equilibrium in the game $\tilde{\Gamma}^{\zeta}$. (66)

c) Definition (66) is equivalent to the following apparent strengthening of Definition 10:

A symmetric $\zeta \in \Delta(S \times S)$ is a perfect correlated equilibrium if $(\delta^{id},\delta^{id})$ satisfies Condition (A′) (equivalently, (B′)). (67)

Proof:

(Part a) Definitions 10 and 12 are equivalent because the function h given in (62) satisfies property (64) and h maps $\mathring{\Delta}(T)$ onto $\mathring{Q}$. Moreover, the continuous function g given in (77) satisfies (78) and the property that $g(\mathring{Q}) \subseteq \mathring{\Delta}(T)$.[14]

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[14] To prove the equivalence between Definitions 10 and 12, the details are as follows. Consider any symmetric $\zeta \in \Delta(S \times S)$.

First, let $\delta \in D$ satisfy Condition (A′), and let sequence $(\delta_1^n,\delta_2^n) \in \mathring{Q} \times \mathring{Q}$ be as given in Condition (A′). We choose $x = g(\delta) \in \Delta(T)$ and choose the sequence $(x_1^n,x_2^n) = (g(\delta_1^n),g(\delta_2^n)) \in \mathring{\Delta}(T) \times \mathring{\Delta}(T)$. Then, $x \in h^{-1}(D)$ and $(x_1^n,x_2^n) \to (x,x)$. Moreover, for all $j \in \{1,2\}$, x is a best reply to x_j^n over $\Delta(T)$. Thus, (x,x) is a perfect equilibrium in $\tilde{\Gamma}^{\zeta}$.

Second, let $x \in h^{-1}(D)$ be such that (x,x) is a perfect equilibrium in $\tilde{\Gamma}^{\zeta}$. Then, there exists a sequence $(x_1^n,x_2^n) \in \mathring{\Delta}(T) \times \mathring{\Delta}(T)$ such that $(x_1^n,x_2^n) \to (x,x)$ and for all $j \in \{1,2\}$, x is a best reply to x_j^n over $\Delta(T)$. We choose $\delta = h(x) \in Q$ and choose the sequence $(\delta_1^n,\delta_2^n) = (h(x_1^n),h(x_2^n)) \in$

(Part b) As the finite normal form game $\tilde{\Gamma}^{\zeta}$ is a two-player game, for any $x \in \Delta(T)$, (x,x) is a perfect equilibrium if and only if (x,x) is a Nash equilibrium and x is undominated (i.e., not weakly dominated by any strategy $y \in \Delta(T)$). (Cf. Weibull (1995), Proposition 1.4). Clearly, δ^{id} is undominated and $(\delta^{id}, \delta^{id})$ is a Nash equilibrium if and only if some $x \in h^{-1}(D)$ satisfies the same property, if and only if every $x \in h^{-1}(D)$ satisfies the same property.
(Part c) This can be proved along the lines of the proof of part (a). Q.E.D.

Definition (67) is the same as Dhillon and Mertens's (1996) definition of a perfect direct correlated equilibrium distribution when it is used in our symmetric context. Therefore, our notion of a perfect correlated equilibrium is the same as their notion of a perfect direct correlated equilibrium distribution.[15]

Definition 13 A *strongly proper correlated equilibrium* is a symmetric $\zeta \in \Delta(S \times S)$ such that there is an $x \in h^{-1}(D)$ such that (x,x) is a proper equilibrium in the game $\tilde{\Gamma}^{\zeta}$.

Proposition 11 *If $\zeta \in \Delta(S \times S)$ is a strongly proper correlated equilibrium, then it is a proper correlated equilibrium.*

Proof: It suffices to show the following:
Claim: Let $\varepsilon > 0$ and $x, y \in \overset{\circ}{\Delta}(T)$. Suppose that for all $\delta', \delta'' \in T$:

$$U^{\zeta}(\delta', y) < U^{\zeta}(\delta'', y) \Rightarrow x(\delta') \le \varepsilon x(\delta''). \tag{68}$$

Then, for every $s, s', s'' \in S$,

$$U^{\zeta}(s', h(y) \mid s) < U^{\zeta}(s'', h(y) \mid s) \Rightarrow (h(x))(s' \mid s) \le \varepsilon (h(x))(s'' \mid s). \tag{69}$$

To prove the Claim, consider any $s, s', s'' \in S$. Define T_{-s} as the set of functions $\delta_{-s} : S \setminus \{s\} \to S$. For every δ_{-s}, define (s', δ_{-s}) as the element in T that agrees with δ_{-s} over $S \setminus \{s\}$ and assigns s' at s. The elements (s'', δ_{-s}) are defined similarly. Then, based on (51), for every $\delta_{-s} \in T_{-s}$, we have the following:

$$U^{\zeta}((s', \delta_{-s}), y) - U^{\zeta}((s'', \delta_{-s}), y) = U^{\zeta}(s', h(y) \mid s) - U^{\zeta}(s'', h(y) \mid s). \tag{70}$$

$\overset{\circ}{Q} \times \overset{\circ}{Q}$. Then, $\delta \in D$ and $(\delta_1^n, \delta_2^n) \to (\delta, \delta)$. Moreover, for all $j \in \{1,2\}$, δ is a best reply to δ_j^n over Q. Thus, δ satisfies Condition (A$'$).

[15] Dhillon and Mertens's definition of a perfect correlated equilibrium is not restricted only to direct correlation mechanisms. Thus, their general notion differs from ours.

Now, we suppose that $U^{\zeta}(s',h(y)\,|\,s) < U^{\zeta}(s'',h(y)\,|\,s)$. Then, we have $U^{\zeta}((s', \delta_{-s}),y) < U^{\zeta}((s'',\delta_{-s}),y)$. Based on (68), we have

$$x((s',\delta_{-s})) \leq \varepsilon x((s'',\delta_{-s})) \quad \text{for all} \quad \delta_{-s} \in T_{-s}. \tag{71}$$

Then, based on the definition (62) of h, we have

$$(h(x))(s'\,|\,s) = \sum_{\delta_{-s}\in T_{-s}} x((s',\delta_{-s})) \leq \sum_{\delta_{-s}\in T_{-s}} \varepsilon x((s'',\delta_{-s})) = \varepsilon (h(x))(s''\,|\,s). \tag{72}$$

Thus, (69) holds. This proves the Claim.

Then, Proposition 11 follows.[16] Q.E.D.

For the Nash equilibrium, if $x \in \Delta(S)$ is an ESS, then (x,x) is a proper equilibrium. (Cf. Weibull (1995), p. 42, Proposition 2.4.) The following result shows that the same is true for an ESC and a strongly proper correlated equilibrium.

Proposition 12 *If* ζ *is an ESC, then it is a strongly proper correlated equilibrium.*

Proof: We apply Lemma 6 as given in the Appendix. Although it is stated in terms of the game G, it can be applied to any two-player symmetric finite normal form game, including $\tilde{\Gamma}^{\zeta}$.

For the game $\tilde{\Gamma}^{\zeta}$, we consider the set $h^{-1}(D) \in \Delta(T)$. We prove that $h^{-1}(D)$ satisfies conditions (i) and (ii), as given Lemma 6.

As D is nonempty, closed, and convex, $h^{-1}(D)$ is also nonempty, closed, and convex. As ζ is an ESC, $h^{-1}(D)$ is contained in the set of symmetric Nash equilibrium strategies in $\tilde{\Gamma}^{\zeta}$. Thus, $h^{-1}(D)$ satisfies condition (i).

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[16] To establish Proposition 11 from the Claim, the details are as follows. Let symmetric ζ be a strongly properly correlated equilibrium. Then, there exists an $x \in h^{-1}(D)$ such that (x,x) is a proper equilibrium in $\tilde{\Gamma}^{\zeta}$. That is, there exists a sequence $\varepsilon^n \in (0,1)$ and a sequence $(x_1^n,x_2^n) \in \mathring{\Delta}(T)\times\mathring{\Delta}(T)$ such that $\varepsilon^n \to 0$, $(x_1^n,x_2^n)\to(x,x)$, and for every n and every $i,j\in\{1,2\}$ with $i \neq j$,

$$U^{\zeta}(\delta',x_j^n) < U^{\zeta}(\delta'',x_j^n) \Rightarrow x_i^n(\delta') \leq \varepsilon^n x_i^n(\delta'') \qquad \forall \delta',\delta'' \in T. \tag{$*$}$$

Now, we define $\delta = h(x) \in D$, and define the sequence $(\delta_1^n,\delta_2^n) = (h(x_1^n),h(x_2^n)) \in \mathring{Q}\times\mathring{Q}$. Then, $(\delta_1^n,\delta_2^n)\to(\delta,\delta)$. Based on ($*$) and the Claim, for every $i,j\in\{1,2\}$ with $i\neq j$, and every $s,s',s''\in S$, we have:

$$U^{\zeta}(s',\delta_j^n\,|\,s) < U^{\zeta}(s'',\delta_j^n\,|\,s) \Rightarrow \delta_i^n(s'\,|\,s) \leq \varepsilon^n \delta_i^n(s''\,|\,s).$$

That is, (54) holds. Thus, (δ,δ) satisfies Condition (C′). Hence, ζ is a proper correlated equilibrium.

Claim: There is an $\overline{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon})$, all $x \in h^{-1}(D)$ and all $y \in \Delta(T)$:

$$\text{Supp}(y) \subseteq B((1-\varepsilon)x + \varepsilon y) \Rightarrow (1-\varepsilon)x + \varepsilon y \in h^{-1}(D), \tag{73}$$

where $\text{Supp}(y) = \{\delta \in T : y(\delta) > 0\}$, and $B(z) = \{\delta \in T : U^{\zeta}(\delta, z) = \max_{\delta' \in T} U^{\zeta}(\delta', z)\}$ for all $z \in \Delta(T)$.

To prove the Claim, based on Lemma 3a, we can choose any uniform invasion barrier $\overline{\varepsilon} > 0$. Consider any $\varepsilon \in (0, \overline{\varepsilon})$, and any $x \in h^{-1}(D)$ and $y \in \Delta(T)$. We suppose that $\text{Supp}(y) \subseteq B((1-\varepsilon)x + \varepsilon y)$. Then, $U^{\zeta}(y, (1-\varepsilon)x + \varepsilon y) = \max_{w \in \Delta(T)} U^{\zeta}(w, (1-\varepsilon)x + \varepsilon y)$. Hence, $U^{\zeta}(h(y), (1-\varepsilon)h(x) + \varepsilon h(y)) = \max_{\delta \in Q} U^{\zeta}(\delta, (1-\varepsilon)h(x) + \varepsilon h(y))$. As $x \in h^{-1}(D)$, we have $U^{\zeta}(h(y), (1-\varepsilon)\delta^{id} + \varepsilon h(y)) = \max_{\delta \in Q} U^{\zeta}(\delta, (1-\varepsilon)\delta^{id} + \varepsilon h(y))$. As $\overline{\varepsilon}$ is a uniform invasion barrier and $\varepsilon < \overline{\varepsilon}$, we have $h(y) \in D$, thus $y \in h^{-1}(D)$. As $h^{-1}(D)$ is convex, $(1-\varepsilon)x + \varepsilon y \in h^{-1}(D)$. This establishes the Claim. Thus, $h^{-1}(D)$ also satisfies condition (ii).

Based on Lemma 6, there exists an $x \in h^{-1}(D)$ such that (x, x) is a proper equilibrium in $\tilde{\Gamma}^{\zeta}$. That is, ζ is a strongly proper correlated equilibrium. Q.E.D.

Based on Propositions 11 and 12 if ζ is an ESC, then it is a proper correlated equilibrium. Hence, it is also a perfect correlated equilibrium based on Proposition 9. However a proper or perfect correlated equilibrium need not be an ESC. In Example 5, the completely mixed Nash equilibrium (x, x) is a proper equilibrium, where $x = (1/3)X + (1/3)Y + (1/3)Z$. Hence, $\zeta = x \times x$ is a proper correlated equilibrium. However it is not an ESC.

Whether the notion of strong properness is strictly stronger than that of properness in correlated equilibria remains an open question.

IV. Concluding Remarks

The conventional ESS approach focuses only on the Nash equilibrium, wherein nature provides private and independent signals to each player. Although the ESS yields a good selection of Nash equilibria, it is restrictive in the sense that in many cases people observe public and correlated signals. To deal with these cases, the concept of the correlated equilibrium is a natural extension of the Nash equilibrium.

In this paper, we provided an evolutionary approach to the correlated equilibrium as a natural selection criterion. Our concept of the ESC captures evolutionary stability for both direct and indirect mechanisms that generate a correlated equilibrium. We have filled the existing gap between the ESS and correlated equilibria and demonstrated that our new concept of the ESC is a generalization of the ESS. We also suggested other refinements to the ESC that have corresponding analogues in the refinements of Nash equilibria. For this purpose, we examined a perfect correlated equilibrium and a proper correlated equilibrium.

V. Appendix

5.1. A Linear Mapping from $\Delta(T)$ onto Q

In section 3, we use the linear function $h:\Delta(T)\rightarrow Q$ defined by (62). We also use the properties of h given in the following lemma.

Lemma 5 *For the linear* $h:\Delta(T)\rightarrow Q$ *defined by (62), the following is true:*

(a) $h(\Delta(T))=Q$*,*

(b) $h(\mathring{\Delta}(T))=\mathring{Q}$.

Proof: Consider any $\delta\in Q$. For each $\hat{\delta}\in T$, we define

$$\lambda_{\hat{\delta}}=\Pi_{s\in S}\delta(\hat{\delta}(s)\,|\,s)\,. \tag{74}$$

Clearly, $\lambda_{\hat{\delta}}\geq 0$. It can also be proved that

$$\text{a) } \sum_{\hat{\delta}\in T}\lambda_{\hat{\delta}}=1\,,$$

$$\text{b) } \delta(s'\,|\,s)=\sum_{\hat{\delta}\in\{\hat{\delta}\in T:\hat{\delta}(s)=s'\}}\lambda_{\hat{\delta}} \quad \text{for all } s,s'\in S\,. \tag{75}$$

Thus,

$$\delta=h(x)\,,\quad \text{where } x=\sum_{\hat{\delta}\in T}\lambda_{\hat{\delta}}\hat{\delta}\in\Delta(T)\,. \tag{76}$$

Hence, $h(\Delta(T))=Q$. This proves Part (a) of the lemma.

For Part (b), based on definition (62), it is clear that $h(\mathring{\Delta}(T))\subseteq\mathring{Q}$. Conversely, consider any $\hat{\delta}\in\mathring{Q}$. Based on (74), it is clear that $\lambda_{\hat{\delta}}>0$ for all $\hat{\delta}\in T$. Then, based on (76) we have $\delta=h(x)$, where $x=\sum_{\hat{\delta}\in T}\lambda_{\hat{\delta}}\hat{\delta}\in\mathring{\Delta}(T)$. Thus, $\mathring{Q}\subseteq h(\mathring{\Delta}(T))$. Hence, Part (b) of the lemma follows. Q.E.D.

As noted in (76), the continuous function $g:Q\rightarrow\Delta(T)$ defined by

$$g(\delta)=\sum_{\hat{\delta}\in T}\lambda_{\hat{\delta}}\hat{\delta} \tag{77}$$

satisfies

$$h(g(\delta))=\delta \quad \text{for all} \quad \delta\in Q. \tag{78}$$

Moreover, based on (77), we also have $g(\overset{\circ}{Q})\subseteq\overset{\circ}{\Delta}(T)$.

5.2. Proof of Lemma 3

(Part (a)) We choose the linear function h mapping $\Delta(T)$ onto Q as given in (62).

For the given ESC ζ, we define $Z\subseteq\Delta(T)$ as the union of all of the boundary faces of $\Delta(T)$ that do not contain any element of the closed convex set $h^{-1}(D)$, where $D=\{\delta'\in Q:\delta'\simeq^{\zeta}\delta^{id}\}$. Then, for all $x\in Z$, there is an $\bar{\varepsilon}>0$ such that

$$U^{\zeta}(\delta^{id},\varepsilon h(x)+(1-\varepsilon)\delta^{id})>U^{\zeta}(h(x),\varepsilon h(x)+(1-\varepsilon)\delta^{id})$$
$$\text{for all} \quad \varepsilon\in(0,\bar{\varepsilon}). \tag{79}$$

As Z is compact, it follows that we can take that invasion barrier $\bar{\varepsilon}$ as uniform overall $x\in Z$.[17] That is, we can fix an $\bar{\varepsilon}>0$ such that

$$\text{(79) holds for all} \quad x\in Z. \tag{80}$$

Now, we show that this uniform barrier $\bar{\varepsilon}$ also applies to those $x\in\Delta(T)\setminus h^{-1}(D)$. To prove this, it suffices to prove that

$$\text{(79) holds for all} \quad x\in\Delta(T)\setminus(h^{-1}(D)\cup Z). \tag{81}$$

[17] This can be proved by an argument similar to that used by Weibull (1995), p. 44, proof of Proposition 2.5, paragraph 2, as follows. First, for every $\delta\in h(Z)$, and every $\varepsilon\in\mathbb{R}$, we define

$$f(\varepsilon,\delta)=A_{\delta}+\varepsilon J_{\delta},$$

where $A_{\delta}=U^{\zeta}(\delta^{id}-\delta,\delta^{id})$ and $J_{\delta}=U^{\zeta}(\delta^{id}-\delta,\delta-\delta^{id})$. By (22a), $A_{\delta}\geq 0$. By (22b), if $A_{\delta}=0$, then $J_{\delta}>0$. Then, we define a function $b:h(Z)\to(0,1]$ by

$$b(\delta)=1 \quad \text{for} \quad J_{\delta}\geq 0,$$
$$=\min\left\{-\frac{A_{\delta}}{J_{\delta}},1\right\} \quad \text{for} \quad J_{\delta}<0.$$

It can be verified that $f(\varepsilon,\delta)>0$ for all $\delta\in h(Z)$ and all $\varepsilon\in(0,b(\delta))$. Moreover, it is easy to prove that b is continuous. As $h(Z)$ is compact, $\bar{\varepsilon}=\min_{\delta\in h(Z)}b(\delta)>0$. This $\bar{\varepsilon}$ satisfies (80).

To prove (81), we require
Claim 1: Let $T' \subseteq T$. Then, for all $x \in \Delta(T') \setminus (h^{-1}(D) \cup Z)$,

$$\text{there exists a } \lambda \in (0,1)\text{, an } x^1 \in \Delta(T') \cap Z\text{,}$$
$$\text{and an } x^2 \in \Delta(T') \cap h^{-1}(D) \text{ such that } x = \lambda x^1 + (1-\lambda)x^2. \tag{82}$$

Proof of Claim 1 is given below.

Now, consider any $x \in \Delta(T) \setminus (h^{-1}(D) \cup Z)$. Based on Claim 1 (with $T' = T$), we can choose a $\lambda \in (0,1)$, an $x^1 \in Z$, and an $x^2 \in h^{-1}(D)$ such that $x = \lambda x^1 + (1-\lambda)x^2$. We consider any $\varepsilon \in (0,\overline{\varepsilon})$. Based on (80), we have

$$U^{\zeta}(\delta^{id}, \varepsilon h(x^1) + (1-\varepsilon)\delta^{id}) > U^{\zeta}(h(x^1), \varepsilon h(x^1) + (1-\varepsilon)\delta^{id}). \tag{83}$$

As $h(x^2) \in D$, we have

$$U^{\zeta}(\delta^{id}, \varepsilon h(x^1) + (1-\varepsilon)\delta^{id}) = U^{\zeta}(h(x^2), \varepsilon h(x^1) + (1-\varepsilon)\delta^{id}). \tag{84}$$

Therefore,

$$\begin{aligned} &U^{\zeta}(\delta^{id}, \varepsilon h(x^1) + (1-\varepsilon)\delta^{id}) \\ &\quad > U^{\zeta}(\lambda h(x^1) + (1-\lambda)h(x^2), \varepsilon h(x^1) + (1-\varepsilon)\delta^{id}) \\ &\quad = U^{\zeta}(h(x), \varepsilon h(x^1) + (1-\varepsilon)\delta^{id}). \end{aligned} \tag{85}$$

Furthemore,

$$\begin{aligned} &U^{\zeta}(\delta^{id}, \varepsilon h(x^2) + (1-\varepsilon)\delta^{id}) \\ &\quad = U^{\zeta}(\delta^{id}, \delta^{id}) \quad (\text{as } h(x^2) \in D) \\ &\quad \geq U^{\zeta}(h(x), \delta^{id}) \quad (\text{by (22a)}) \\ &\quad = U^{\zeta}(h(x), \varepsilon h(x^2) + (1-\varepsilon)\delta^{id}) \quad (\text{as } h(x^2) \in D). \end{aligned} \tag{86}$$

Then, based on (85) and (86), we have

$$U^{\zeta}(\delta^{id}, \varepsilon h(x) + (1-\varepsilon)\delta^{id}) > U^{\zeta}(h(x), \varepsilon h(x) + (1-\varepsilon)\delta^{id}). \tag{87}$$

This establishes (81), which together with (80), implies that the uniform invasion barrier $\overline{\varepsilon}$ holds for all $x \in \Delta(T) \setminus h^{-1}(D)$, i.e., (79) holds for all $x \in \Delta(T) \setminus h^{-1}(D)$. As h maps $\Delta(T)$ onto Q, it follows that $\overline{\varepsilon}$ is a uniform barrier for all $\delta' \in Q$ with $\delta' \neq^{\zeta} \delta^{id}$. This establishes Part (a) of Lemma 3.

(Part (b)) We prove Part (b) based on our proof of Part (a).

We choose a uniform barrier $\overline{\varepsilon} > 0$ as given in Part (a) of the lemma.

Claim 2: There exists a neighborhood N of D in Q satisfying

$$N \subseteq \{\varepsilon\delta^1 + (1-\varepsilon)\delta^2 \,|\, (\delta^1 \in Q \setminus D) \,\&\, (\delta^2 \in D) \,\&\, (\varepsilon \in [0, \overline{\varepsilon}))\}. \tag{88}$$

Proof of Claim 2 is given below.

We choose a neighborhood N of D, as given in Claim 2. Then, $\delta^{id} \in D \subseteq N$.

Consider any $\delta' \in N$ with $\delta' \neq^{\zeta} \delta^{id}$. We claim that

$$U^{\zeta}(\delta^{id}, \delta') > U^{\zeta}(\delta', \delta'). \tag{89}$$

First, there exists a $\delta^1 \in Q \setminus D$, $\delta^2 \in D$, and $\varepsilon \in (0, \overline{\varepsilon})$ such that $\delta' = \varepsilon\delta^1 + (1-\varepsilon)\delta^2$. As $\delta^1 \neq^{\zeta} \delta^{id}$, we have

$$U^{\zeta}(\delta^{id}, \varepsilon\delta^1 + (1-\varepsilon)\delta^{id}) > U^{\zeta}(\delta^1, \varepsilon\delta^1 + (1-\varepsilon)\delta^{id}). \tag{90}$$

As $\delta^2 \simeq^{\zeta} \delta^{id}$, it follows that

$$\begin{aligned} &U^{\zeta}(\delta^{id}, \varepsilon\delta^1 + (1-\varepsilon)\delta^2) = U^{\zeta}(\delta^{id}, \varepsilon\delta^1 + (1-\varepsilon)\delta^{id}) \\ &U^{\zeta}(\delta^1, \varepsilon\delta^1 + (1-\varepsilon)\delta^2) = U^{\zeta}(\delta^1, \varepsilon\delta^1 + (1-\varepsilon)\delta^{id}). \end{aligned} \tag{91}$$

Thus,

$$U^{\zeta}(\delta^{id}, \delta') > U^{\zeta}(\delta^1, \delta'). \tag{92}$$

Again, as $\delta^2 \simeq^{\zeta} \delta^{id}$, we have

$$U^{\zeta}(\delta^{id}, \delta') = U^{\zeta}(\delta^2, \delta'). \tag{93}$$

Then, based on (92) and (93), we have

$$U^{\zeta}(\delta^{id}, \delta') > U^{\zeta}(\varepsilon\delta^1 + (1-\varepsilon)\delta^2, \delta'). \tag{94}$$

That is, (89) holds. This establishes Part (b) of the Lemma 3. Q.E.D.

Proof of Claim 1. We prove Claim 1 by induction over $n = \#(T')$.

Claim 1 is clearly true for all $T' \subseteq T$ with $\#(T') = 1$.

Suppose Claim 1 is true for all $T'' \subseteq T$ with $\#(T'') = n$. We consider any $T' \subseteq T$ with $\#(T') = n+1$. We prove that Claim 1 is also true for T'. There are two cases:

(Case 1) Suppose $\Delta(T') \cap h^{-1}(D) = \emptyset$. As $\delta^{id} \in h^{-1}(D)$, we have $\delta^{id} \notin T$. As such, $T' \subset T$, and $\Delta(T')$ is a boundary face of $\Delta(T)$. Thus, $\Delta(T') \subseteq Z$. Hence, Claim 1 holds for T'.

(Case 2) Suppose $\Delta(T') \cap h^{-1}(D) \neq \emptyset$. We consider any $x \in \Delta(T') \setminus (h^{-1}(D) \cup Z)$. There are two cases:

(Case 2a) Suppose $x \notin \overset{\circ}{\Delta}(T')$. Then, $x \in \text{bdy}(T')$, where $\text{bdy}(\Delta(T')) = \Delta(T') \setminus \overset{\circ}{\Delta}(T') = \cup_{T'' \subset T} \Delta(T'')$. So, $x \in \Delta(T'')$ for some $T'' \subset T'$. Based on the induction hypothesis, Claim 1 is true for T''. As such, there exists a $\lambda \in (0,1)$, an $x^1 \in \Delta(T'') \cap Z$, and an $x^2 \in \Delta(T'') \cap h^{-1}(D)$ such that $x = \lambda x^1 + (1-\lambda)x^2$. As $T'' \subseteq T$, x satisfies (82).

(Case 2b) Suppose $x \in \overset{\circ}{\Delta}(T')$. We choose any $y \in \Delta(T') \cap h^{-1}(D)$. Then, we can choose an $\alpha \in (0,1)$ and a $w \in \text{bdy}(\Delta(T'))$ such that $w = x + ((1-\alpha)/\alpha)(x-y)$. Thus, $x = \alpha w + (1-\alpha)y$. As $x \notin h^{-1}(D)$, $y \in h^{-1}(D)$, and $h^{-1}(D)$ is convex, we have $w \notin h^{-1}(D)$. There are two additional cases.

(Case 2b.i) Suppose $w \in Z$. Then, x satisfies (82) with $\lambda = \alpha$, $x^1 = w$ and $x^2 = y$.

(Case 2b.ii) Suppose $w \notin Z$. As $w \in T''$ for some $T'' \subset T'$ and based on the induction hypothesis, there exists a $\beta \in (0,1)$, a $w^1 \in \Delta(T'') \cap Z$, and a $w^2 \in \Delta(T'') \cap h^{-1}(D)$ such that $w = \beta w^1 + (1-\beta)w^2$. Then, we have $x = \lambda x^1 + (1-\lambda)x^2$, where $\lambda = \alpha\beta \in (0,1)$, $x^1 = w^1 \in \Delta(T') \cap Z$, and $x^2 = \gamma w^2 + (1-\gamma)y \in \Delta(T') \cap h^{-1}(D)$, and $\gamma = \alpha(1-\beta)/(1-\alpha\beta) \in (0,1)$. Hence, x satisfies (82).

Thus, we show that Claim 1 holds for T'.

Hence, by induction, Claim 1 holds for all $T' \subseteq T$. Q.E.D.

Proof of Claim 2. We provide a neighborhood N of D in Q that satisfies (88).

First, for every $x \in \Delta(T)$, we define

$$\tau(x) = \min\{\lambda \in [0,1] : x = \lambda x^1 + (1-\lambda)x^2 \text{ for some } x^1 \in Z \text{ and } x^2 \in h^{-1}(D)\}. \tag{95}$$

As Z and $h^{-1}(D)$ are compact, by Claim 1 (with $T' = T$), $\tau(x)$ is well-defined for all $x \in \Delta(T)$. Clearly, $\tau(x) = 0$ for all $x \in h^{-1}(D)$.

We claim that we can choose a real number $\alpha > 0$ such that the set

$$\tilde{N} = \{x \in \Delta(T) : \|y - x\| < \alpha \quad \text{for some} \quad y \in h^{-1}(D)\} \tag{96}$$

satisfies

$$\tau(x) < \bar{\varepsilon} \quad \text{for all} \quad x \in \tilde{N}\,. \tag{97}$$

Suppose no such an α exists. Then, there exists a sequence $x_n \in \Delta(T)$, a sequence $x_n^1 \in Z$, a sequence $x_n^2 \in h^{-1}(D)$, a sequence $\lambda_n \in [\bar{\varepsilon}, 1]$, and a sequence $y_n \in h^{-1}(D)$ such that for all n:

$$\begin{aligned} x_n &= \lambda_n x_n^1 + (1-\lambda_n) x^2, \\ \|x_n - y_n\| &\le 1/n\,. \end{aligned} \tag{98}$$

By taking convergent subsequences if necessary, we can assume that $x_n - x, y_n \to x, x_n^1 \to x^1, x_n^2 \to x^2$, and $\lambda_n \to \lambda$ for some $x, x^2 \in h^{-1}(D)$, $x^1 \in Z$, and $\lambda \in [\bar{\varepsilon}, 1]$. Then,

$$x = \lambda x^1 + (1-\lambda) x^2\,. \tag{99}$$

So,

$$\delta = \lambda \delta^1 + (1-\lambda)\delta^2\,, \tag{100}$$

where $\delta = h(x) \in D$, $\delta^1 = h(x^1) \notin D$, and $\delta^2 = h(x^2) \in D$. Hence,

$$K^{\zeta}(\delta, \delta^{id}) = \lambda K^{\zeta}(\delta^1, \delta^{id}) + (1-\lambda) K^{\zeta}(\delta^2, \delta^{id})\,. \tag{101}$$

That is,

$$\zeta = \lambda K^{\zeta}(\delta^1, \delta^{id}) + (1-\lambda)\zeta\,. \tag{102}$$

As $\lambda > 0$, we have $\zeta = K^{\zeta}(\delta^1, \delta^{id})$. Then, by Lemma 2 we have $\delta^1 \in D$. This contradicts the fact that $\delta^1 \notin D$. This establishes that we can choose a real number $\alpha > 0$ such that the set $\tilde{N}$ defined by (96) satisfies (97).

Based on (97), for all $x \in \tilde{N}$, we have

$$x = \lambda x^1 + (1-\lambda) x^2 \quad \text{for some} \quad \lambda \in [0, \bar{\varepsilon}), \quad x^1 \in Z \quad \text{and} \quad x^2 \in h^{-1}(D)\,. \tag{103}$$

Now, we choose the continuous function $g: Q \to \Delta(T)$ defined by (77). We define the set

$$N = g^{-1}(\tilde{N}). \tag{104}$$

By (78), we have $D = h(g(D))$. So, $g(D) \subseteq h^{-1}(D) \subseteq \tilde{N}$. Hence, $D \subseteq g^{-1}(\tilde{N})$ $= N$. As $\tilde{N}$ is open in $\Delta(T)$ and g is continuous, N is a neighborhood of D in Q.

It remains to show that N satisfies property (88). Consider any $\delta \in N$. We choose the element $x = g(\delta) \in \tilde{N}$, so $\delta = h(x)$. By (103), $x = \varepsilon x^1 + (1-\varepsilon)x^2$ for some $\varepsilon \in [0, \bar{\varepsilon})$, $x^1 \in Z$, and $x^2 \in h^{-1}(D)$. Hence, $\delta = \varepsilon\delta^1 + (1-\varepsilon)\delta^2$, where δ^1 $= h(x^1) \notin D$, and $\delta^2 = h(x^2) \in D$. This establishes that N satisfies (88). Q.E.D.

5.3. Proof of Equivalence between Conditions A′ and B′

We consider any $\delta \in D$ and prove that δ satisfies Condition A′ if and only if it satisfies Condition B′.

Condition B′ implies Condition A′. Let Condition B′ hold. We choose a sequence $(\varepsilon_1^n, \varepsilon_2^n) \in P^S \times P^S$ and a sequence $(\delta_1^n, \delta_2^n) \in Q_{\varepsilon_1^n} \times Q_{\varepsilon_2^n}$ as given there.

Then, $(\delta_1^n, \delta_2^n) \in \mathring{Q} \times \mathring{Q}$.

We now prove that (δ_1^n, δ_2^n) satisfies (48) for all large n. Consider any $i, j \in \{1,2\}$ with $i \neq j$. As $\delta_i^n \to \delta$ and $\varepsilon_i^n \to 0$, for all large n, and all $s \in S$ with $\sum_{s' \in S} \zeta(s, s') > 0$, we have

$$\delta(s' \mid s) > 0 \Rightarrow \delta_i^n(s' \mid s) > \varepsilon_{i,s,s'}^n \qquad \forall s' \in S, \tag{105}$$

where $\varepsilon_i^n = (\varepsilon_{i,s,s'}^n)_{s,s' \in S} \in P^S$. Based on (49) and (51),

$$\delta_i^n(s' \mid s) > \varepsilon_{i,s,s'}^n \Rightarrow U^{\zeta}(s', \delta_j^n \mid s) = \max_{s'' \in S} U^{\zeta}(s'', \delta_j^n \mid s) \qquad \forall s' \in S. \tag{106}$$

Then, based on Lemma 4, (48) holds for all large n.

By taking away the initial terms if necessary, we can assume that the sequence (δ_1^n, δ_2^n) satisfies (48) for all n.

Thus, we establish Condition A′.

Condition A′ implies Condition B′. Let Condition A′ hold. We choose a sequence $(\delta_1^n, \delta_2^n) \in \mathring{Q} \times \mathring{Q}$ as given there.

We choose any sequence $\alpha^n \in (0,1)$ with $\alpha^n \to 0$. For each $i \in \{1,2\}$ and each n, we define a perturbation $\varepsilon_i^n = (\varepsilon_{i,s,s'}^n)_{s,s' \in S} \in P^S$ by

$$\varepsilon_{i,s,s'}^n = \alpha^n \delta_i^n (s' \mid s) \qquad \forall s, s' \in S . \tag{107}$$

Then, for all $s \in S$, it is easy to verify that

$$\Delta_{\varepsilon_{i,s}^n}(S) = \{ (1-\alpha^n) x + \alpha^n \delta_i^n (s) : x \in \Delta(S) \} , \tag{108}$$

where $\varepsilon_{i,s}^n = (\varepsilon_{i,s,s'}^n)_{s' \in S} \in P$. Then,

$$Q_{\varepsilon_i^n} = \{ (1-\alpha^n) \delta' + \alpha^n \delta_i^n : \delta' \in Q \} . \tag{109}$$

For each n, and each $i \in \{1,2\}$, we define the assignment:

$$\tilde{\delta}_i^n = (1-\alpha^n) \delta + \alpha^n \delta_i^n . \tag{110}$$

Then, based on (109) $\tilde{\delta}_i^n \in Q_{\varepsilon_i^n}$. Furthermore, based on (110), we have $(\tilde{\delta}_1^n, \tilde{\delta}_2^n) \to (\delta, \delta)$.

We claim that for each n and each $i, j \in \{1,2\}$ with $i \neq j$:

$$U^{\zeta}(\tilde{\delta}_i^n, \tilde{\delta}_j^n) = \max_{\delta' \in Q_{\varepsilon_i^n}} U^{\zeta}(\delta', \tilde{\delta}_j^n) . \tag{111}$$

First, as δ is a best reply to δ_j^n over Q, based on (109) and (110), $\tilde{\delta}_i^n$ is a best reply to δ_j^n over $Q_{\varepsilon_i^n}$, i.e.,

$$U^{\zeta}(\tilde{\delta}_i^n, \delta_j^n) = \max_{\delta' \in Q_{\varepsilon_i^n}} U^{\zeta}(\delta', \delta_j^n) . \tag{112}$$

Furthermore, as $(\delta_1^n, \delta_2^n) \to (\delta, \delta)$, based on (48), δ is a best reply to δ over Q. Then, again based on (109) and (110), $\tilde{\delta}_i^n$ is a best reply to δ over $Q_{\varepsilon_i^n}$, i.e.,

$$U^{\zeta}(\tilde{\delta}_i^n, \delta) = \max_{\delta' \in Q_{\varepsilon_i^n}} U^{\zeta}(\delta', \delta) . \tag{113}$$

As $\tilde{\delta}_j^n = (1-\alpha^n) \delta + \alpha^n \delta_j^n$, based on (112) and (113), it follows that (111) holds.

Thus, we obtain a sequence $(\varepsilon_1^n, \varepsilon_2^n) \in P^S \times P^S$ and a sequence $(\tilde{\delta}_1^n, \tilde{\delta}_2^n) \in Q_{\varepsilon_1^n} \times Q_{\varepsilon_2^n}$ satisfying the property required in Condition B′. Q.E.D.

5.4. Proof of Proposition 8

We choose the functions $y \mapsto \delta_y$ and $\delta \mapsto y_\delta$ as given in (55) and (56). We have

$$\text{a) } \delta_y \in \overset{\circ}{Q} \quad \forall y \in \overset{\circ}{\Delta}(S),$$
$$\text{b) } y_\delta \in \overset{\circ}{\Delta}(S) \quad \forall \delta \in \overset{\circ}{Q}. \tag{114}$$

Based on (57), we have

$$\text{a) } U^\zeta(\delta_y, \delta_{\tilde{y}}) = V(y, \tilde{y}) \quad \forall y, \tilde{y} \in \Delta(S).$$
$$\text{b) } U^\zeta(\delta, \tilde{\delta}) = V(y_\delta, y_{\tilde{\delta}}) \quad \forall \delta, \tilde{\delta} \in Q. \tag{115}$$

Furthermore based on definitions (55), (56), and (50), it follows that for all $s, s' \in S$ we have

$$\text{a) } U^\zeta(s', \delta_y \mid s) = x(s)V(s', y) \quad \forall y \in \Delta(S).$$
$$\text{b) } U^\zeta(s', \delta \mid s) = x(s)V(s', y_\delta) \quad \forall \delta \in Q. \tag{116}$$

Proof of Part (a) ("Only if" Part) Suppose (x,x) is a perfect equilibrium. As (x,x) satisfies Condition A, we can choose a sequence $(x_1^n, x_2^n) \in \overset{\circ}{\Delta}(S) \times \in \overset{\circ}{\Delta}(S)$ that converges to (x,x) and satisfies (44). We choose the sequence (δ_1^n, δ_2^n) where $\delta_i^n = \delta_{x_i^n}$.

As (x_1^n, x_2^n) converges to (x,x) and the mapping $y \to \delta_y$ is continuous, we have $(\delta_1^n, \delta_2^n) \to (\delta_x, \delta_x)$. It is easy to verify that $K^\zeta(\delta_x, \delta^{id}) = \zeta$. That is, $\delta_x \in D$.

We now show that (δ_x, δ_x) satisfies Condition A′. First, based on (114), we have $(\delta_1^n, \delta_2^n) \in \overset{\circ}{Q} \times \overset{\circ}{Q}$. Second, to see (48), we consider any $j \in (1,2\}$. For every n, and every $\delta' \in Q$, we have

$$U^\zeta(\delta_x, \delta_j^n) = V(x, x_j^n) \geq V(y_{\delta'}, x_j^n) = U^\zeta(\delta', \delta_j^n). \tag{117}$$

Thus, (48) holds (with $\delta = \delta_x$). As (δ_x, δ_x) satisfies Condition A′, ζ is a perfect correlated equilibrium.

("If" Part) Suppose that ζ is a perfect correlated equilibrium. Then, we can choose a $\delta \in D$ and a sequence (δ_1^n, δ_2^n) as given in Condition A′. We choose the sequence (x_1^n, x_2^n) where $x_i^n = y_{\delta_i^n}$.

As $(\delta_1^n, \delta_2^n) \to (\delta, \delta)$ and the mapping $\delta \to y_\delta$ is continuous, we have

$(x_1^n, x_2^n) \to (y_\delta, y_\delta)$. Note that

$$x \times x = \zeta = K^\zeta(\delta, \delta) = y_\delta \times y_\delta \,. \tag{118}$$

Therefore, $(y_\delta, y_\delta) = (x, x)$.

We now show that (x, x) satisfies Condition A. First, based on (114), we have $(x_1^n, x_2^n) \in \mathring{\Delta}(S) \times \mathring{\Delta}(S)$. Second, to see (44), consider any $j \in \{1,2\}$. For every n , and every $y' \in \Delta(S)$, we have

$$V(x, x_j^n) = U^\zeta(\delta, \delta_j^n) \geq U^\zeta(\delta_{y'}, \delta_j^n) = V(y', x_j^n) \,. \tag{119}$$

Thus, (44) holds (with $x_i = x$). As (x, x) satisfies Condition A, it is a perfect equilibrium.

Proof of Part (b) ("Only if" Part) Suppose (x, x) is a proper equilibrium. As (x, x) satisfies Condition C, we can choose a sequence $\varepsilon^n \in (0,1)$ with $\varepsilon^n \to 0$, and choose a sequence $(x_1^n, x_2^n) \in \mathring{\Delta}(S) \times \mathring{\Delta}(S)$ with $(x_1^n, x_2^n) \to (x, x)$, which satisfies (46). We choose the sequence (δ_1^n, δ_2^n) where $\delta_i^n = \delta_{x^n}$. As shown in the proof of "only if" in Part (a), we have: $(\delta_1^n, \delta_2^n) \in \mathring{Q} \times \mathring{Q}$, $(\delta_1^n, \delta_2^n) \to (\delta_x, \delta_x)$, and $\delta_x \in D$.

We must prove (54). Consider any $i, j \in \{1,2\}$ with $i \neq j$, any n and any $s \in S$.

(Case 1) Suppose $x(s) = 0$. Then, $U^\zeta(s', \delta_j^n \mid s) = 0 = U^\zeta(s'', \delta_j^n \mid s)$ for all $s', s'' \in S$. Thus, (54) holds.

(Case 2) Suppose $x(s) > 0$. Then, for all $s', s'' \in S$:

$$\begin{aligned} U^\zeta(s', \delta_j^n \mid s) < U^\zeta(s'', \delta_j^n \mid s) &\Rightarrow V(s', x_j^n) < V(s'', x_j^n) \quad \text{(by (116a))} \\ &\Rightarrow x_i^n(s') \leq \varepsilon^n x_i^n(s'') \quad \text{(by (46))} \\ &\Rightarrow \delta_i^n(s' \mid s) \leq \varepsilon^n \delta_i^n(s'' \mid s) \quad \text{(by (55))}. \end{aligned} \tag{120}$$

Thus, we prove (54). As (δ_x, δ_x) satisfies Condition C′, ζ is a proper correlated equilibrium.

("If" Part) Suppose that ζ is a proper correlated equilibrium. Then, we can choose a $\delta \in D$ such that (δ, δ) satisfies Condition C′. We choose a sequence $\varepsilon^n \in (0,1)$ with $\varepsilon^n \to 0$, and a sequence $(\delta_1^n, \delta_2^n) \in \mathring{Q} \times \mathring{Q}$ with $(\delta_1^n, \delta_2^n) \to (\delta, \delta)$, and such that (54) holds. We choose the sequence (x_1^n, x_2^n) , where $x_i^n = y_{\delta_i^n}$.

As shown in the proof of "if" in Part (a), we have $(x_1^n, x_2^n) \in \mathring{\Delta}(S) \times \mathring{\Delta}(S)$ and

$(x_1^n, x_2^n) \to (x, x)$.

We must prove (46). Consider any $i, j \in \{1,2\}$ with $i \neq j$, any n, and any $s, s' \in S$. Suppose that $V(s', x_j^n) < V(s'', x_j^n)$. Based on (116b), we have

$$U^{\zeta}(s', \delta_j^n \mid s) < U^{\zeta}(s'', \delta_j^n \mid s) \quad \text{for all } s \in S \text{ with } x(s) > 0. \tag{121}$$

Based on (54), we have

$$\delta_i^n(s' \mid s) \leq \varepsilon^n \delta_i^n(s'' \mid s) \quad \text{for all } s \in S \text{ with } x(s) > 0. \tag{122}$$

Then, based on definition (56),

$$x_i^n(s') = \sum_{s \in S} x(s) \delta_i^n(s' \mid s) \leq \sum_{s \in S} \varepsilon^n x(s) \delta(s'' \mid s) = \varepsilon^n x_i^n(s''). \tag{123}$$

Thus, (46) holds. As (x, x) satisfies Condition C, it is a proper equilibrium. Q.E.D.

5.5. Statement and Proof of Lemma 6

We used the following lemma in our proof of Proposition 12. It provides a sufficient condition for a set Θ to contain a symmetric proper equilibrium strategy.

Lemma 6 *Consider the two-player symmetric finite normal form game* G. *Let* $\Theta \subseteq \Delta(S)$ *satisfy*

(i) Θ *is nonempty, closed, and convex, and is contained in the set of symmetric Nash equilibrium strategies.*

(ii) There is an $\bar{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0, \bar{\varepsilon})$, *all* $x \in \Theta$ *and all* $y \in \Delta(S)$:

$$\text{Supp}(y) \subseteq B((1-\varepsilon)x + \varepsilon y) \Rightarrow (1-\varepsilon)x + \varepsilon y \in \Theta, \tag{124}$$

where $\text{Supp}(y) = \{s \in S : y(s) > 0\}$ *and* $B(z) = \{s \in S : V(s, z) = \max_{s' \in S} V(s', z)\}$ *for all* $z \in \Delta(S)$.

Then there exists an $\bar{x} \in \Theta$ *such that* $(\bar{x}, \bar{x})$ *is a proper equilibrium.*

Lemma 6 is a "symmetric" variant of Theorem 5 of Swinkels (1992), which yields a proper equilibrium for an equilibrium evolutionarily stable (EES) set. We modify Swinkels's result in several aspects: (a) Our set Θ is contained in the set of mixed strategies of single players, and Swinkels's ESS set is contained in the set of mixed strategy profiles of all players. (b) Our set Θ is assumed to be convex, and

Swinkels's ESS set is proved to be convex. (c) Our game is symmetric and we yield a symmetric proper equilibrium, and Swinkels's game and proper equilibrium are not necessarily symmetric. Taking these features into account, we can easily modify Swinkels's proof and establish Lemma 6.

For the reader's convenience, we provide a proof here.

Proof of Lemma 6

(Step 1 Two claims.) Define:

$$\Lambda = \{z \in \Delta(S) : B(z) \subseteq B(x) \quad \text{for some} \quad x \in \Theta.\} \tag{125}$$

Along the lines of Swinkels's proofs,[18] we can easily obtain the following variants of his Lemmas 9 and 10.

Claim 1: Let $z \in \Lambda$, $y \in \Delta(X)$, $x \in \Theta$, and $\alpha \in [0,1]$ be such that $z = (1-\alpha)x + \alpha y$. Then,

$$\text{Supp}(y) \subseteq B(z) \Rightarrow z \in \Theta . \tag{126}$$

Claim 2: There exists an $\bar{\varepsilon} > 0$ such that for all $x \in \Theta$ and all $z \in \Delta(S)$,

$$\|x - z\| \leq \bar{\varepsilon} \Rightarrow z \in \Lambda . \tag{127}$$

We choose such an $\bar{\varepsilon}$ as given in Claim 1 and define

$$Y = \{y \in \Delta(S) : \max_{x \in \Theta} \|y - x\| \leq \bar{\varepsilon}\} . \tag{128}$$

Based on Claim 2, $Y \subseteq \Lambda$.

(Step 2: Obtaining a symmetric ε-proper equilibrium.) For each $\varepsilon \in (0,1)$, we define a correspondence $P_\varepsilon : \Delta(S) \to \Delta(S)$ by $P_\varepsilon(x) = \{y \in \Delta(S) \mid y$ satisfies (129)$\}$,

$$\begin{aligned} &\text{a)} \quad y(s) \geq \varepsilon^{\#(S)} / \#(S) \qquad \forall s \in S , \\ &\text{b)} \quad V(s,x) > V(t,x) \Rightarrow y(t) \leq \varepsilon y(s) \qquad \forall s,t \in S . \end{aligned} \tag{129}$$

As P_ε is nonempty-valued, convex-valued, and upper hemicontinuous, it has a fixed point. For any sequence $\varepsilon^n \to 0$, any sequence x^n of fixed points P_{ε^n}, and

[18] This requires the following variant of his Lemma 7: If $x, y \in \Delta(S)$ is such that $B(x) \cap B(y) \neq \emptyset$, then $B((1-\lambda)x + \lambda y) = B(x) \cap B(y)$ for all $\lambda \in (0,1)$. This result can be easily obtained along the lines of Swinkels's proof.

any $x \in \Delta(S)$, if $x^n \to x$, then (x,x) is a proper equilibrium.

(Step 3: Defining a retraction $A : \Delta(S) \to Y$*.)* For each $y \in \Delta(S)$, define:

$$W(y) = \arg\min_{x \in \Theta} \|x - y\|$$
$$L(y) = \{z \in Y : z = (1-\alpha)W(y) + \alpha y \quad \text{for some} \quad \alpha \in [0,1]\}$$
$$A(y) = \arg\min_{z \in L(y)} \|z - y\| . \tag{130}$$

Using an argument similar to that given by Swinkels (1992), p. 327, paragraph 3, it follows that A is single-valued, continuous, and maps $\Delta(S)$ onto Y, and that $A|_Y$ is an identity mapping, i.e., $A : \Delta(S) \to Y$ is a retraction.

(Step 4: Obtaining a proper equilibrium $(\bar{x},\bar{x})$ *with* $\bar{x} \in \Theta$*.)* For each $\varepsilon \in (0,1)$, we define a correspondence $R_\varepsilon : \Delta(S) \to \Delta(S)$ by:

$$R_\varepsilon(x) = P_\varepsilon(A(x)) . \tag{131}$$

Then, R_ε has a fixed point for each $\varepsilon \in (0,1)$. We can choose any sequence $\varepsilon^n \to 0$, any sequence x^n of fixed points R_{ε^n}, and any $\bar{x} \in \Delta(S)$ such that $x^n \to \bar{x}$.

Now, we apply the argument used by Swinkels (1992), p. 327, paragraph 4, and prove that $\bar{x} \in \Theta$. First, for each n, we have $x^n \in P_{\varepsilon^n}(A(x^n))$. As such:

$$V(s, A(x^n)) > V(t, A(x^n)) \Rightarrow x^n(t) \le \varepsilon^n x(s) \qquad \forall s,t \in S . \tag{132}$$

As $\varepsilon^n \to 0$ and $x^n \to \bar{x}$, we have

$$\text{Supp}(\bar{x}) \subseteq B(A(\bar{x})) . \tag{133}$$

Moreover, $A(\bar{x}) = (1-\alpha)W(\bar{x}) + \alpha\bar{x}$ for some $\alpha \in [0,1]$. Furthermore, $W(\bar{x}) \in \Theta$ and $A(\bar{x}) \in Y \subseteq \Lambda$. Then, based on Claim 1 we have $A(\bar{x}) \in \Theta$. Hence, $A(\bar{x}) = \bar{x}$.

As $\bar{x} \in \Theta$ and $x^n \to \bar{x}$, for all large n we have $x^n \in Y$. As such, $A(x^n) = x^n$, and thus $x^n \in P_{\varepsilon^n}(x^n)$. Hence, $(\bar{x},\bar{x})$ is a proper equilibrium. Q.E.D.

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